THE POLISH MATHEMATICIAN, JOSEPH HOENE-WRONSKI A BRIEF BIO-SKETCH AND SOME PROPERTIES OF WRONSKIAN

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Bio-Sketch

The Polish Mathematician:
JÓZEF MARIA HOENE-WROŃSKI

Uncompromising philosopher, hard worker. His motto was "*Learn from great people great things which they have taught us. Their weaknesses are of secondary importance*".

Born in Wolsztyn—-August 23, 1779. Father—a Czech emigrant.

Jozef attended school in Poznan—1786-1790.

Bio-Sketch

Joined army against father's will. Father's opposition was great, but the boy's determination was even greater.
In 1792 he ran away from home and changed name to make father's search more difficult.
During the defence of Warsaw against the Prussian army he commanded a battery and was awarded a medal by commander-in-chief T. Kosciuszko.
Having taken to captivity, served Russian army (1795-97).

Bio-Sketch

Sudden death of his father changed Wronski's plans, inherited a lot of money, used it to devote himself to studies.
Quit army and travelled west. In 1800, visited England, and then came to France. Became a member of Marseilles' Academy of Science.

Bio-Sketch

Underwent an enlightenment on August 15, 1803 at a ball on Napolean's birthday. Claimed that he understood the mystery of the universe, decided to reform human thought and create a universal philosophical system. Inspired by this revelation, took the name of Maria and became known as Jozef Maria Hoene-Wronski. During financial hardship, he used to do tuition. He was impressed by the aptitude and intelligence of one of his students and married her.

Bio-Sketch

Posed three problems of discovering relation between matter and energy (note Wronski's incredibly deep insight), and formation of universe from celestial objects.

Bio-Sketch

Singled out two aspects of mathematics, namely Theories and Algorithmic techniques. Fascinated by continued fractions, Wronski considered the problem of interpolation of a function by continued fractions.

Bio-Sketch

In 1810, he sent the article "On the fundamental principles of algorithmic methods" to the French Academy. The Academy (Lagrange was a member) had mixed (mostly negative) opinions and did not accept it. Wronski withdrew his paper directing bitter words towards the academics from Paris using phrases like "Born Enemies OF Truth"

Bio-Sketch

During 1814-1819, Wronski published: *Philosophy of Infinity, Philosophy of algorithmic techniques and Criticism of Laplace's generating functions*.
In 1826, Wronski went to Belgium. Belgian scientists were the first to bring Wronski into worldwide scientific literature.

Bio-Sketch

Wronski has not only created a philosophical system, but also its applications to politics, history, economy, law, psychology, music and education. In 1829, published a treatise on steam engines.

Bio-Sketch

His iron nature required little sleep and food, begins work early in the morning and only after a couple of hours of work he would have a meal saying: "Now I have earned my day". The seriousness of his work and struggle against misfortune did not spoil his calm personality and cheerful character.

Bio-Sketch

He considered adversaries of his philosophy as enemies of the truth, stating his arguments vehemently like "These gentlemen are interested neither in progress nor in the truth".

Bio-Sketch

Wronski's financial situation became worse, dressed in worn out clothes, asked Napolean for help but was denied. Fortunately he met a wealthy banker and Merchant P. Arson who agreed to help him on the condition that Wronski would reveal the Secret of the Absolute. After some time Arson asked Wronski to reveal the secret, however Wronski refused. The matter went to a court, and Wronski won.

Bio-Sketch

In 1853, wronski sent two papers to the Navy department who said that Laplace's formulas are sufficient for the needs of the Navy. This was a severe blow for the 75-year old scientist who, after 50 years of hard work, once more did not find recognition. He died on August 9, 1853 in Neuilly. Before his death he whispered to his wife "*Lord Almighty, I had so much more to say*".

Motivation OF This Talk

THESE UNFORTUNATE AND UNFAIR EVENTS IN THE LIFE OF WRONSKI MOTIVATED ME TO STUDY WRONSKIANS (THE DETERMINANT HE DISCOVERED IN 1882) AND RECOGNIZE HIS SIGNIFICANT CONTRIBUTION TO MATHEMATICS.

Wronskian

For n real functions $f_1, ..., f_n$ of x, that are $n-1$ times differentiable on an interval I, their Wronskian is defined by the determinant $W(f_1, \ldots, f_n) =$

$$\begin{vmatrix} f_1 & \ldots & f_n \\ f_1' & \ldots & f_n' \\ . & \ldots & . \\ . & \ldots & . \\ . & \ldots & . \\ f_1^{(n-1)} & \ldots & f_n^{(n-1)} \end{vmatrix}$$

Wronskian

So, the Wronskian of two differentiable functions f and g is the determinant $W(f, g) = fg' - gf'$. If the the functions $f_1, \ldots, f_n$ are linearly dependent, then the Wronskian vanishes. Therefore, the functions are linearly independent on I if their Wronskian W does not vanish identically on I.

Wronskian

We note the following result of Peano [1] wich says that, if $W = 0$ for all $x \in I$, then the functions may or may not be linearly dependent. For example, the functions x^2 and $x|x|$ have continuous derivatives and their Wronkian vanishes everywhere, but are not linearly dependent in any neighborhood of 0.

Peano [1] showed that, if the functions are analytic, then the vanishing of Wronskian in an interval implies linear dependence.

Wronskian

As the identical vanishing of the Wronskian of two analytic functions implies their linear dependence, it is interesting to examine analytic functions whose Wronskians are constants (not necessarily zero). Examples of such functions in pairs are $(cosx, sinx)$, (e^{x}, e^{-x}), and $(1, x)$.These pairs of functions are solutions of the following second order linear homogeneous differential equations $y'' + y = 0$(simple harmonic motion), $y'' - y = 0$ and $y'' = 0$ respectively. This is supported by the Abel's identity: $W' = -p(x)W$, where W is the Wronskian of the solutions of the equation $y'' + p(x)y + q(x)y = 0$. So, if W is constant, then $p(x) = 0$.

Results

We consider such functions with constant Wronskians and prove the following result.

Theorem

Let f, g, h be analytic functions on an interval I such that $W(f, g) = k_1$, $W(g, h) = k_2$ and $W(h, f) = k_3$ where k_1, k_2, k_3 are constants. Then f, g, h are linearly dependent and related by

$$k_1 h + k_2 g + k_3 g = 0. \tag{1}$$

Results

Its proof is based on the following two lemmas.

Lemma 1.

$$\frac{d}{dx}W(f,g) = \begin{vmatrix} f & g \\ f'' & g'' \end{vmatrix}$$

The proof is obvious.

Results

Lemma 2. The Wronskian satisfies the Jacobi identity:

$$W(W(f,g),h) + W(W(g,h),f) + W(W(h,f),g) = 0. \quad (2)$$

for three functions f, g, h.
Though this result is known, as pointed out by Poinsot [2], we would like to include its proof for the sake of completeness.

Proof

A direct computation shows that

$$W(W(f,g),h) = fg'h' - gf'h' - hfg'' + hgg'' \qquad (3)$$

Permuting f, g, h twice in the above equation and then adding the three equations yields equation (2) which proves the lemma.

Wronskian of Power Functions

Let us begin with two powers of x, namely x^p and x^q. Their Wronkian is

$$W(x^p, x^q) = \begin{vmatrix} x^p & x^q \\ px^{p-1} & qx^{q-1} \end{vmatrix} = (q-p)x^{p+q-1}$$

Wronskian of Power Functions

Next, we consider three powers, namely x^p, x^q, x^r. Their Wronskian is

$$W(x^p, x^q, x^r) = \begin{vmatrix} x^p & x^q & x^r \\ px^{p-1} & qx^{q-1} & rx^{r-1} \\ p(p-1)x^{p-2} & q(q-1)x^{q-2} & r(r-1)x^{r-2} \end{vmatrix}$$

Using the properties of determinants we simplify it as

$$W(x^p, x^q, x^r) = x^{p+q+r-3}(q-p)(r-p)(r-q) \qquad (4)$$

Wronskian of Power Functions

Similarly, for n powers $x^{p_1}, ..., x^{p_n}$, the patterns for 2 and 3 powers show that

$$\begin{aligned} W(x^{p_1}, ..., x^{p_n}) &= x^{p_1 + ... p_n - n(n-1)/2}(p_2 - p_1)(p_3 - p_1)(p_3 - p_2) \\ ... & \quad (p_n - p_1)(p_n - p_2)...(p_n - p_{n-1}) \qquad (5) \end{aligned}$$

From this, it follows straightaway that

$$W(1, x, x^2, x^3, ..., x^{n-1}) = (1!)(2!)(3!)(4!)....(n-1)! \qquad (6)$$

Wronskian of power Functions

We can also conclude from the above equation that

$$\frac{W(1, x, x^2, x^3, ..., x^n)}{W(1, x, x^2, x^3, ..., x^{n-1})} = n!$$

Concluding Remark

Obviously the Wronskian of two functions is skew-symmetric in its arguments, $W(f, g) = -W(g, f)$. In addition, it also satisfies the Jacobi identity:

$$W(W(f, g), h) + W(W(g, h), f) + W(W(h, f), g) = 0$$

as stated in Lemma 2. Thus the space of all smooth functions on an open interval forms a Lie algebra under Wronskian as the Lie bracket operation.

Concluding Remark

A standard example of a Lie algebra is the 3-dimensional space with cross product of vectors as the Lie bracket, as the cross product is skew symmetric: $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$, and satisfies the Jacobi identity: $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} + (\mathbf{b} \times \mathbf{c}) \times \mathbf{a} + (\mathbf{c} \times \mathbf{a}) \times \mathbf{b} = \mathbf{0}$.

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THANK YOU

THANK YOU FOR YOUR AMAZING PATIENCE.