Volumes and their boundaries in $\mathbb{R}^n$

Joseph Kolibal
The University of New Haven

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The topic

Volumes and their boundaries in $\mathbb{R}^n$

Abstract

The relationship of integrals on the surface of a region in $\mathbb{R}^n$ to integrals over the volume of the region is a fundamental part of calculus. We examine this relationship from the fundamental theorem of calculus and integration by parts, to the theorems of vector calculus, taking note of some interesting aspects of the technique and the diversity of results which can be obtained using these relationships.

In particular, the recursive nature of integration by parts is one of the building blocks of modern mathematics. It is a feature of so many problems that it deserves consideration for inclusion into Paul Halmos’ table of the mathematical elements.

Accordingly,

No doubt many mathematicians have noted that there are some basic ideas that keep cropping up in widely different parts of their subject, combining and re-combining with one another in a way faintly reminiscent of how all matter is made up of elements. A subconscious intuitive awareness of these “elements” of mathematics probably contributes to (possibly it constitutes) the research insight that distinguishes great mathematicians from ordinary mortals. – Paul Halmos$	extsuperscript{1}$

In this presentation, we examine the role of surface and volume and show how these are recurring themes tied to the derivative and integral.

We begin with a simple example, the circle and sphere.

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1Paul R. Halmos was born in Budapest, Hungary, on March 3, 1916. At the age of 15 he enrolled at the University of Illinois to study chemical engineering and later switched to mathematics and philosophy. He received his PhD in 1938. Halmos believed that mathematics is art, and that mathematicians are artists.
Let $S_n$ denote the surface area of an $n$-sphere, and $V_n$ denote the volume or the content of an $n$-sphere.

- The 2-sphere, i.e., the circle of radius $R$ has circumference, i.e., surface $S_2 = 2\pi R$, and its area, or content is $V_2 = \pi R^2$.
- The surface area of a 3-sphere of radius $R$ is $S_3 = 4\pi R^2$, and its volume is $V_3 = (4/3)\pi R^3$.
- For the 2-sphere, $S_2 = \frac{d}{dr} V_2(r)$ since $\frac{d}{dr} \pi R^2 = 2\pi R$.
- For the 3-sphere, $S_3 = \frac{d}{dr} V_3(r)$ since $\frac{d}{dr} (4/3)\pi R^3 = 4\pi R^2$.

Does it generalize?

Assume that the volume of the $n$-sphere is proportional to the radius, then $V_n = \alpha_n r^n$, where $\alpha_n$ is our constant of proportionality for each dimension.

Then by using concentric shells, $dV_n = S_n dr$,

$$S_n = \frac{dV_n}{dr} = n\alpha_n r^{n-1},$$

and it becomes obvious why $S_n$ is the derivative of $V_n$ for each $n$-sphere.

We can do more. Let's evaluate $S_n$.

This means finding the value of $\alpha_n$.

We begin by considering the integral of $e^{-r^2}$ in rectangular, i.e., Cartesian and polar coordinates over all of $\mathbb{R}^2$.

- A clever trick based on a technique evaluating the value of the Gaussian or normal probability density function is motivational: In polar coordinates we have,

$$\int_{\mathbb{R}^2} e^{-x^2-y^2} dA = \int_0^\infty \int_0^{2\pi} e^{-r^2} r dr d\theta = 2\pi \left[ -\frac{1}{2} e^{-r^2} \right]_0^\infty = \pi.$$

While in Cartesian coordinates we have

$$\int_{\mathbb{R}^2} e^{-x^2-y^2} dA = \int_{\mathbb{R}^2} e^{-x^2} e^{-y^2} dx dy = \left( \int_{-\infty}^{\infty} e^{-x^2} dx \right) \left( \int_{-\infty}^{\infty} e^{-y^2} dy \right).$$

Thus, comparing these we conclude $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$, i.e., that

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}.$$
Computing the integrals over \( \mathbb{R}^n \)

Volumes and their boundaries in \( \mathbb{R}^n \)

(i) The Cartesian case in \( \mathbb{R}^n \):

\[
\int_{\mathbb{R}^n} e^{-r^2} dV_n = \left( \int_{-\infty}^{\infty} e^{-x^2} dx \right)^n = \pi^{n/2}
\]

(ii) The spherical case in \( \mathbb{R}^n \):

\[
\int_{\mathbb{R}^n} e^{-r^2} dV_n = \int_0^{\infty} e^{-t} S_n dr = n \alpha_n \int_0^{\infty} e^{-t} r^{n-1} dr
\]

where we substituted for \( S_n \) from (1)

\[
= \frac{n \alpha_n}{2} \int_0^{\infty} e^{-t} t^{n/2 - 1} dt,
\]

on substituting \( r^2 = t \), \( dt = 2r dr \).

\[
= \frac{n \alpha_n}{2} \Gamma(n/2),
\]

where the Gamma function is defined as \( \Gamma(p) = \int_0^{\infty} e^{-t} t^{p-1} dt \).

Putting it together

Volumes and their boundaries in \( \mathbb{R}^n \)

- Since (i) and (ii) are the same volume, we must have that

\[
\frac{n \alpha_n}{2} \Gamma(n/2) = \pi^{n/2} \quad \Rightarrow \quad \alpha_n = \frac{2 \pi^{n/2}}{n \Gamma(n/2)}.
\]

- This means that for the \( n \)-sphere we have

\[
V_n = \frac{2 \pi^{n/2}}{n \Gamma(n/2)} r^n, \quad S_n = \frac{2 \pi^{n/2}}{n \Gamma(n/2)} r^{n-1}.
\]

The size of the sphere in \( \mathbb{R}^n \)

Concentric spheres

- The Gamma function is the generalized factorial function, i.e., for \( n \in \mathbb{N}, \Gamma(n) = (n-1)! = (n-1) \cdot (n-2) \cdot (n-3) \cdot \ldots \cdot 3 \cdot 2 \cdot 1 \).

Thus, we have, for example

\[
\Gamma(1) = 0! = 1, \Gamma(2) = 1! = 1, \Gamma(3) = 2! = 2, \Gamma(4) = 3! = 6, \Gamma(5) = 4! = 24,\ldots
\]

Tabulating these with \( n \) gives interesting insight into the shape of \( \mathbb{R}^n \).
<table>
<thead>
<tr>
<th>n</th>
<th>$V_n/c_n$</th>
<th>$V_n/C_n$</th>
<th>$S_n$</th>
<th>Comments</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2.0000</td>
<td>1.000000</td>
<td>2.0000</td>
<td>Since $S_1 = \frac{2\pi^{1/2}}{\Gamma(1/2)}$, $\Gamma(1/2) = \pi^{1/2}$</td>
</tr>
<tr>
<td>2</td>
<td>3.1416</td>
<td>0.785398</td>
<td>6.2832</td>
<td>Maximum $V_2$. Holds the most $n$-cubes.</td>
</tr>
<tr>
<td>3</td>
<td>4.1888</td>
<td>0.523599</td>
<td>12.566</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>4.9348</td>
<td>0.308425</td>
<td>19.739</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>5.2638</td>
<td>0.164493</td>
<td>26.319</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>5.1677</td>
<td>0.080746</td>
<td>31.006</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>4.7248</td>
<td>0.036912</td>
<td>33.074</td>
<td>Maximum $S_7$ at $n = 7.257$.</td>
</tr>
<tr>
<td>8</td>
<td>4.0587</td>
<td>0.015854</td>
<td>32.497</td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>3.2985</td>
<td>0.006442</td>
<td>29.687</td>
<td></td>
</tr>
</tbody>
</table>

Note that $V_n/C_n \to 0$, i.e., spheres are vanishingly small inside of cubes in $\mathbb{R}^n$ for large $n$.

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**Calculus and boundaries**

- The Fundamental Theorem of Calculus states\(^2\) that:
  - Let $f$ be an integrable function on $[a,b]$. For $x \in [a,b]$, let $F(x) = \int_{a}^{x} f(t)\,dt$. Then $F$ is continuous on $[a,b]$, and $F'(x)$ exists and equals $f(x)$ at every $x$ at which $f$ is continuous.
  
  Let $F$ be continuous on $[a,b]$ and differentiable except at finitely many points in $[a,b]$, and let $f$ agree with $F'(x)$ where $F'$ is defined. If $f$ is integrable on $[a,b]$, then $\int_{a}^{b} f(t)\,dt = F(b) - F(a)$.

- The point to emphasize here is that the area or content of $f$ on $[a,b]$ can be obtained, under suitably mild conditions, using only the endpoints of the interval $[a,b]$, i.e., using only the boundary of $[a,b]$ and some suitably constructed anti-derivative $F$.

  There appears to be a fundamental link between volumes and surfaces that are their boundaries.

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**Integration by parts**

- Since the definite integral depends on the values of $f$ on the boundary of the domain, it makes sense to consider techniques that can use this result to some advantage, most notably integration by parts.

- The evaluation of integrals involving many elementary functions is achieved through integration by parts.

  We examine this topic in greater detail.
Integrating by parts

▶ Given \( f : [a, b] \rightarrow \mathbb{R} \) and \( g : [a, b] \rightarrow \mathbb{R} \), continuously differentiable, then from the product rule we have

\[
(f(x)g(x))' = f(x)g'(x) + g(x)f'(x).
\]

▶ Integrating both sides gives

\[
\int (f(x)g(x))' \, dx = \int f(x)g'(x) \, dx + \int g(x)f'(x) \, dx
\]

\[
f(x)g(x) = \int f(x)g'(x) \, dx + \int g(x)f'(x) \, dx
\]

On rearranging,

\[
\int f(x)g'(x) \, dx = f(x)g(x) - \int f(x)'g(x) \, dx
\]

▶ Integrating both sides over \([a, b]\) gives the definite integrals

\[
\int_a^b f(x)g'(x) \, dx = f(x)g(x)|_a^b - \int_a^b f(x)'g(x) \, dx
\]

Example

▶ We know that for \( D \equiv (d/dx) \), that \( Dx^p = px^{p-1} \), for \( p \in \mathbb{Z} \), with

\[
\ldots, Dx^2 = 2x^1, Dx^1 = 1, Dx^0 = 0, Dx^{-1} = -x^{-2}, \ldots
\]

▶ Note that there is a hole in the results, with no power of \( x \) having a derivative to a -1 power, i.e., \( Dx^p \neq cx^{-1} \) for any \( x^p, p \in \mathbb{Z} \).

▶ Of course, \( D \ln x = x^{-1} \), so that

\[
\ldots, Dx^2 = 2x^1, Dx^1 = 1, Dx^0 = 0, D \ln x = x^{-1}, Dx^{-1} = -x^{-2}, \ldots
\]

▶ Taking integrals, i.e., \( I \equiv \int (.) \, dx \), of the functions in this chain in (2) gives,

\[
\ldots, Ix^2 = \frac{x^3}{3}, Ix^1 = \frac{x^2}{2}, Ix^0 = x, I \ln x = \ldots, Ix^{-1} = \ln x, \ldots
\]

We need to fill the hole in the new chain.

Integrating \( \ln x \)

▶ We find the answer by integration by parts, letting

\[
f(x) = \ln x, \text{ and } g'(x) = 1.
\]

▶ Then since \( \int g'(x) \, dx = \int dx = x \), and \( f'(x) = (\ln x)' = 1/x \), we have

\[
\int \ln x(1) \, dx = (\ln x)(x) - \int x \left( \frac{1}{x} \right) \, dx
\]

\[
= x \ln x - \int dx
\]

\[
= x \ln x - x + C.
\]

Alright, nice technique, but why is it special?

Recursion!
Examining the Gamma function again Volumes and their boundaries in $\mathbb{R}^n$

- Consider: $\Gamma(p) = \int_0^\infty e^{-t}t^{p-1}dt = -\int_0^\infty d(e^{-t})t^{p-1}$
  
  $$= -e^{-t}t^{p-1}|_0^\infty + \int_0^\infty (t^{p-1})e^{-t}dt$$
  
  $$= 0 + \int_0^\infty (p-1)t^{p-2}e^{-t}dt$$
  
  $$= (p-1)\int_0^\infty t^{p-2}e^{-t}dt$$
  
  $$= (p-1)\Gamma(p-1)$$

- This is similar to $n! = n(n-1)!$ for factorials.
- Note that $\Gamma(1) = \int_0^\infty e^{-t}dt = 1$.

Indeed, for non-negative $x$, the Gamma function has been shown to be the function that satisfies: $\Gamma(1) = 1$ and the recursion that $\Gamma(x+1) = x\Gamma(x)$.

A recursive example Volumes and their boundaries in $\mathbb{R}^n$

- We examine the integral, $I_k = \int_0^\pi \sin^k x dx$.
- Integration by parts gives
  
  $$I_k = \int_0^\pi \sin^k x dx = \int_0^\pi \sin^{k-1} x \cdot \frac{\sin x}{f(x)} dx$$
  
  $$= -\sin^{k-1} x \cos x|_0^\pi + \int_0^\pi (k-1) \cos^2 x \sin^{k-2} x dx$$
  
  $$= -\sin^{k-1} x \cos x|_0^\pi + \int_0^\pi (k-1) \cos^2 x \sin^{k-2} x dx$$
  
  $$= (k-1) \int_0^\pi \sin^{k-2} x dx - (k-1) \int_0^\pi \sin^k x dx$$
  
  $$= (k-1) \int_0^\pi \sin^{k-2} x dx - (k-1) \int_0^\pi \sin^k x dx$$

$$= (k-1) \int_0^\pi \sin^{k-2} x dx - (k-1) \int_0^\pi \sin^k x dx$$

(3)

A recursive example (cont’d) Volumes and their boundaries in $\mathbb{R}^n$

- Based on the definition of $I_k$ in (3), we must have
  
  $$I_k = (k-1)I_{k-2} - (k-1)I_k \implies I_k = \frac{k-1}{k}I_{k-2}. \quad (4)$$

- Knowing $I_0$, $I_1$ and then $I_2$, we can find all $I_n$ directly as,
  
  $$I_3 = \frac{3-1}{3}I_0, \quad I_4 = \frac{4-1}{4}I_1, \quad I_5 = \frac{5-1}{5}I_2, \quad I_6 = \frac{6-1}{6}I_3, \ldots$$

since we can easily compute the first three as

$$I_0 = \int_0^\pi dx = \pi, \quad I_1 = \int_0^\pi \sin(x)dx = 2,$$

and $I_2 = \int_0^\pi \sin^2(x)dx = \frac{1}{2}(x - \sin(x)\cos(x)) = \pi,$
- Applying these results leads to the formulae (one for odd, and one for even integers):

\[ I_{2k} = \left( \frac{2k-1}{2k} \right) \left( \frac{2k-3}{2k-2} \right) \cdots \left( \frac{3}{4} \right) \left( \frac{1}{2} \right) \pi, = \pi \prod_{j=1}^{k} \frac{2j-1}{2j} \]

\[ I_{2k+1} = \left( \frac{2k}{2k+1} \right) \left( \frac{2k-2}{2k-1} \right) \cdots \left( \frac{4}{5} \right) \left( \frac{2}{3} \right) 2 = 2 \prod_{j=1}^{k} \frac{2j}{2j+1}. \]

We can prove this using induction.

- Constructing \( \frac{I_{2k}}{I_{2k+1}} = \frac{\pi}{2} \prod_{j=1}^{k} \left( \frac{2j-1}{2j} \right) \left( \frac{2j+1}{2j} \right) \).

We can show that \( \frac{I_{2k}}{I_{2k+1}} \to 1 \) as \( k \to \infty \).

**Wallis’s product**

- Putting it together, we get

\[ \lim_{k \to \infty} \frac{I_{2k}}{I_{2k+1}} = 1 = \frac{\pi}{2} \lim_{k \to \infty} \prod_{j=1}^{k} \left( \frac{2j-1}{2j} \right) \left( \frac{2j+1}{2j} \right) \]

or rearranging,

\[ \frac{\pi}{2} = \prod_{j=1}^{\infty} \frac{2j}{2j-1} \cdot \frac{2j}{2j+1} \]

or alternatively,

\[ \frac{\pi}{2} = \lim_{k \to \infty} 2 \cdot \frac{2}{3} \cdot \frac{4}{5} \cdot \frac{4}{5} \cdot \frac{6}{7} \cdot \frac{6}{7} \cdots \frac{2k-2}{2k-1} \cdot \frac{2k}{2k-1} \cdot \frac{2k}{2k-1} \cdot \frac{2k-2}{2k-1} \cdot \frac{2k}{2k-1} \]

- This is Wallis’ product, written down in 1655 by John Wallis.\(^3\)

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\(^3\)John Wallis was born on November 23, 1616. He received his Bachelor of Arts degree in 1637, and a Master’s in 1640, in mathematics, then entered priesthood, and was later appointed in 1649 as the Savilian Chair of Geometry at Oxford University until his death on October 28, 1703.
<table>
<thead>
<tr>
<th>The vanishing boundary term</th>
<th>Volumes and their boundaries in $\mathbb{R}^n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>In the development of Wallis’ Product, we used integration by parts, obtaining the intermediate result,</td>
<td></td>
</tr>
</tbody>
</table>

$$-\sin^{n-1}x\cos(x)|_0^\pi + \int_0^\pi (n-1)\cos^2 x\sin^{n-2}x\,dx$$

in which the boundary term $-\sin^{n-1}x\cos(x)|_0^\pi$ goes to zero, or vanishes.

- In a sense, the recursion was possible because of this result.

A good result always merits further investigation.

We will pick up on this idea to discuss derivatives of derivatives of functions with kinks and discontinuities, and then bring in some amazing results from vector calculus.

<table>
<thead>
<tr>
<th>Summary</th>
<th>Volumes and their boundaries in $\mathbb{R}^n$</th>
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<tbody>
<tr>
<td>We’ve only touched the surface of exploring the many links in calculus between a domain and its boundary.</td>
<td></td>
</tr>
</tbody>
</table>

- Mathematics is about patterns.
  - Finding them;
  - Working with them; and,
  - Discovering new ones.

- Read, when you have the opportunity: *Mathematics as a Creative Art*, by P.R. Halmos, *American Scientist*, 56(1968), 375-389:

  *Mathematics – this may surprise or shock you some – is never deductive in its creation. The mathematician at work makes vague guesses, visualizes broad generalizations, and jumps to unwarranted conclusions. He arranges and rearranges his ideas, and he becomes convinced of their truth long before he can write down a logical proof.*

  – Paul Halmos (excerpt from *Mathematics as a Creative Art*).

  Make your own art, find your own patterns.

Thank You

*Department of Mathematics at The University of New Haven*

JKolibal@newhaven.edu

http://www.newhaven.edu