The concept has multidimensional extensions. We will get to these.

The Fundamental Theorem of Calculus states that given $f : [a,b] \to \mathbb{R}$ and $g : [a,b] \to \mathbb{R}$, continuously differentiable, we can obtain

$$\int_a^b f'(x)g(x)\,dx = f(x)g(x)|_a^b - \int_a^b f(x)g'(x)\,dx.$$

We examine this formula in greater detail.

The FTC in $\mathbb{R}^n$ states that given $f$, the integral is independent of path, depending only on the starting and ending points, i.e.,

- The concept has multidimensional extensions. We will get to these.
- The evaluation of integrals involving many elementary functions is achieved through integration by parts.

Again, if we examine the formula, the integral is recast into interior terms and boundary terms.

We examine this topic in greater detail.

Integration by parts
Volumes and their boundaries in $\mathbb{R}^n$

- Given $f : [a,b] \to \mathbb{R}$ and $g : [a,b] \to \mathbb{R}$, continuously differentiable, then from the product rule we have

$$f(x)g'(x) = f(x)g(x)' + g(x)f'(x).$$

Integrating both sides gives

$$\int f(x)g'(x)\,dx = \int f(x)g(x)\,dx + \int g(x)f'(x)\,dx.$$

On rearranging,

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int f(x)y'(x)g(x)\,dx.$$

Integration by parts over $[a,b]$ gives the definite integrals

$$\int_a^b f(x)g'(x)\,dx = f(x)g(x)|_a^b - \int_a^b f(x)y'(x)g(x)\,dx.$$

What is the derivative of $|x|$ on $[-1,1]$? It’s well defined for $x \neq 0$.

If $x = 0$ it is not differentiable since the derivative from the left, i.e., $-1$, is not equal to the derivative from the right, i.e., 1.

Perhaps the derivative is the average of these values, i.e., 0.

This would mean some type of averaging would have to be constructed, and so it is natural to consider the integral, which

Indeed, this is exactly what is needed.
The weak derivative $f'$ of a function $f$ is defined as the distribution $\delta$ such that for every compactly supported smooth function $\phi$, we have
\[ \int \phi f' \, dx = -\int f \phi' \, dx. \]

1. **The difficulty with using** $f'$ **is that it is not known.**
2. **Alright, let’s require it to work for any suitable $\phi$.**
3. **Let $f(a, b) \to \mathbb{R}$, and $g(a, b) \to \mathbb{R}$ be local integrable functions.**
4. **We say that $g$ is the weak derivative of $f$ if**
\[ \int_a^b f(x) \phi'(x) \, dx = -\int_a^b g(x) \phi(x) \, dx \]
for all smooth functions $\phi : (a, b) \to \mathbb{R}$ with compact support.

So how does it work?

---

### Example (cont’d)

- **Putting it together by equating these results to the right side of (3)**, we have
  \[
  \text{(LHS)} : \int_{-1}^1 |x| \phi(x) \, dx = \int_{-1}^0 (1) \phi(x) \, dx + \int_0^1 (-1) \phi(x) \, dx
  = -\int_{-1}^1 g(x) \phi(x) \, dx : \text{(RHS)}.
  \]

And so by definition, we must have $g(x) = \begin{cases} -1 & \text{if } x < 0 \\ 1 & \text{if } x > 0 \end{cases}$.

Thus, $g$ is the required weak derivative.

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### Multidimensional calculus

- **The Heaviside function is given by** $H(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x > 0 \end{cases}$.

- Writing $D_1$ for the weak derivative, we can show that $D_1 H(x) = \delta(x)$, where $\delta(x)$ is the fundamental solution of $\Delta$.

So we note that the weak derivative of $f$ is the same as the classical derivative when $f$ is differentiable on $(a, b)$.

It works, and it is not just a tool for evaluating integrals.

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### Integration by parts in $\mathbb{R}^n$

- **Consider functions $f(x)$ and $g(x)$, where $x = (x_1, x_2) \in \mathbb{R}^n$, and examine the partial derivatives,**
\[ \frac{\partial}{\partial x_1} f(x) g_1(x) = f_1(x) g_1(x) + f(x) g_1_1(x), \]
\[ \frac{\partial}{\partial x_2} f(x) g_2(x) = f_2(x) g_2(x) + f(x) g_2_2(x), \]
and adding the left side of (5)–(7) gives
\[ \frac{\partial}{\partial x_1} \frac{\partial}{\partial x_2} \frac{\partial}{\partial x_3} \frac{\partial}{\partial x_4} \frac{\partial}{\partial x_5} g(x) = \nabla \cdot (f(x) \nabla g(x)). \]

- **Adding up the right sides in (5)–(7) gives**
\[ (f_1(x) g_1(x) + f(x) g_1_1(x)) + (f_2(x) g_2(x) + f(x) g_2_2(x)) + (f_3(x) g_3(x) + f(x) g_3_3(x)) = \nabla f(x) \cdot \nabla g(x) + \nabla f(x) \Delta g(x). \]

- **Consequently, equating the left and right sides gives**
\[ \nabla \cdot (f(x) \nabla g(x)) = \nabla f(x) \cdot \nabla g(x) + \nabla f(x) \Delta g(x), \]
and integrating gives
\[ \int_{\Omega} \nabla \cdot (f(x) \nabla g(x)) \, dV = \int_{\Omega} \nabla f(x) \cdot \nabla g(x) \, dV + \int_{\Omega} \nabla f(x) \Delta g(x) \, dV. \]

So how does it work?
### A note on symbolic representation

Volumes and their boundaries in \( \mathbb{R}^n \)

- The preceding example is quite technical and challenging to wade through if the details of vector calculus are not well understood.
- Working with structures in mathematics requires acquiring the ability to manipulate ideas symbolically.

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### The Fundamental Theorem of Calculus

Volumes and their boundaries in \( \mathbb{R}^n \)

- If \( f \) is a continuous function on \([a, b]\) and \( F \) is any antiderivative of \( f \) on \([a, b]\), then
  \[
  \int_a^b f(x) \, dx = F(b) - F(a). \tag{8}
  \]
- The value of the integral depends on the endpoints, not the stuff in-between. Of course, this works because in a sense the stuff in-between is taken care of by how \( F \) is constructed. Thus detail counts.
- All of this is also reminiscent of state functions in thermodynamics, i.e., a property of a system that depends only on the current, equilibrium state of the system. For example, enthalpy and entropy are state functions. Thus, these are properties independent of path, i.e., they depend only on the starting and ending points of the thermodynamic system.

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### Summary

- We've only touched the surface of exploring the many links in calculus between a domain and its boundary.
- Mathematics, and indeed all rational thought, is about patterns. Finding them; Working with them; and, Discovering new ones.
- Read, when you have the opportunity: *Mathematics as a Creative Art*, by P.R. Halmos, American Scientist, 56(1968), 375-389:
  Mathematics – this may surprise or shock you some – is never deductive in its creation. The mathematician at work makes vague guesses, visualizes broad generalizations, and jumps to unwarranted conclusions. He arranges and rearranges his ideas, and he becomes convinced of their truth long before he can write down a logical proof.
  – Paul Halmos (excerpt from *Mathematics as a Creative Art*).

Make your own art, find your own patterns.

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### Thank You

Volumes and their boundaries in \( \mathbb{R}^n \)

Department of Mathematics at The University of New Haven

JKolibal@newhaven.edu

http://www.newhaven.edu