The Logic behind Black–Scholes Formula and Long Term Capital Management

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Options

**Definition**

An option gives one the right, but not the obligation, to buy or sell a share of a security under specified terms. A call option is one that gives the right to buy, and a put option is one that gives the right to sell the security. Both types of options will have an exercise price and an exercise time.

European options can be utilized only at the exercise time, whereas American options can be utilized at any time up to exercise time.

**Assume:** Flat dollars; i.e., zero interest rates, zero inflation.

**Wanted:** The value of an option.
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Simple Example Option

A 100 (dollars) share of stock will be worth either 200 or 50 after one year.

![Figure 1](image)

Let $C$ be a fair price for a *European Call Option with strike price 150*. What is $C$?

**Definition**

A sure-win betting scheme is called an **arbitrage**.

At time 0

- buy $y$ options at $Cy$ (if $y < 0$ sell $-y$ options for $-Cy$) and
- buy $x$ shares at $100x$ (if $x < 0$ sell $-x$ shares for $-100x$).
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![Diagram showing stock outcomes at time 0 and 1]

**Figure: 1**

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\[ 100 \quad \rightarrow \quad 200 \]
\[ 100 \quad \rightarrow \quad 50 \]

\[ t = 0 \quad \text{time} \quad t = 1 \]

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- buy \( y \) options at \( Cy \) (if \( y < 0 \) sell \(-y\) options for \(-Cy\)) and
- buy \( x \) shares at 100\( x \) (if \( x < 0 \) sell \(-x\) shares for \(-100x\)).
Break Even Price of a Call Option

The cost of the transactions is $100x + Cy$ and

$$\text{Value of assets at time 1} = \begin{cases} 200x + 50y & \text{if final share price is 200} \\ 50x & \text{if final share price is 50} \end{cases}$$

Now choose $y$ such that the value of assets are the same at time 1 no matter what the final price of a share.

$$200x + 50y = 50x \quad \Rightarrow \quad y = -3x.$$ 

The minus sign says if $x$ shares of stock is bought at time $t = 0$ then $3x$ call options are sold and the $t = 1$ value of the assets is $50x$. Furthermore

$$\text{Gain} = 50x - (100x + Cy) = 50x - 100x - C(-3x) = -50x + 3Cx.$$ 

At the break even price of $C$ the gain should be 0 and $C = 50/3$. 

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For the previous example, the break even price of the call option was $50/3$. At that price there is no arbitrage, but at any other price there is. If the call option costs more than $50/3$ it is worth your while to sell them. If the cost is less than $50/3$ it is worth your while to buy them.

**Theorem (The Generalized Law of One Price)**

Consider two investments, the first of which costs the fixed amount $C_1$ and the second the amount $C_2$. If $C_1 < C_2$ and the (present value) payoff from the first investment is always at least as large as that from the second investment, then there is arbitrage.

**Proof.**

Arbitrage is obtained by simultaneously buying investment 1 and selling investment 2.
Definition of normal Brownian motion


**Definition**

A collection of random variables, $B(t)$ for $t > 0$ is a **Brownian motion** with drift parameter $\mu$ and variance parameter $\sigma^2$, iff

1. $B(0)$ is a given constant
2. For all $s, t > 0$ the r.v. $B(s + t) - B(s)$ is independent from $B(s)$ and has normal distribution with mean $\mu t$ and variance $\sigma^2 t$.

French mathematician Bachelier, 1900, used Brownian motion to study pricing of stocks.
Brownian motion as the limit of simpler models.

Fix a large number $N$ and let $\Delta \overset{\text{def}}{=} \frac{1}{N}$. For every $i\Delta$ units of time the process, $B_\Delta$ increases by $\sigma \sqrt{\Delta}$ with probability $p = \frac{1}{2} \left(1 + \frac{\mu}{\sigma} \sqrt{\Delta}\right)$ or decreases by $\sigma \sqrt{\Delta}$ with probability $1 - p$. Assume $B_\Delta$ only changes value at positive multiples of $\Delta$.

$$B_{\Delta i}(i\Delta) = B_{\Delta i}(i\Delta) + \sigma \sqrt{\Delta}$$
$$B_{\Delta i}(i\Delta) = B_{\Delta i}(i\Delta) - \sigma \sqrt{\Delta}$$

\begin{align*}
B_{\Delta i}(i\Delta) & \overset{p}{\rightarrow} B_{\Delta i}(i\Delta + \sigma \sqrt{\Delta}) \\
B_{\Delta i}(i\Delta) & \overset{1-p}{\rightarrow} B_{\Delta i}(i\Delta - \sigma \sqrt{\Delta})
\end{align*}

\begin{figure}
\centering
\begin{tikzpicture}
\draw (0,0) -- (4,0) node[midway] {time};
\draw (0,0) -- (0,-1); \node at (0,-1) {$t = i\Delta$};
\draw (4,0) -- (4,-1); \node at (4,-1) {$t = (i+1)\Delta$};
\filldraw[fill=white,draw=black] (0,0) circle (2pt);
\filldraw[fill=white,draw=black] (4,0) circle (2pt);
\end{tikzpicture}
\caption{Figure: 2}
\end{figure}

**Theorem**

As $\Delta \downarrow 0$ one has $B_\Delta \rightarrow B$. 
Geometric Brownian motion

Problems with using Brownian motion to approximate stock prices.

- stock prices are never negative, Brownian motion can be.
- the change in a stock price’s over a fixed amount of time is proportional to that stock’s price - not true for Brownian motion.

One usually models a stock’s price, $S(t)$, with geometric Brownian motion.

**Definition**

Let $B(t)$ be Brownian motion with drift parameter $\mu$ and variance parameter $\sigma^2$. Then $S(t) \overset{\text{def}}{=} e^{B(t)}$ is a geometric Brownian motion process with drift parameter $\mu$ and variance parameter $\sigma^2$. The volatility parameter is $\sigma$. 

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1. $S(t)$ can be written as $se^\tilde{B}(t)$ where $B(0) = \ln(s)$, where $\tilde{B}(0) = 0$ and where both $B(t)$ and $\tilde{B}(t)$ are Brownian motion with the same drift and variance parameters.

2. $E(S) = se^{\mu t + \sigma^2 t/2} = se^{(\mu + \sigma^2/2)t}$ so $\mu + \sigma^2/2$ is the rate of $S(t)$, i.e., the rate the price of the security is growing by.
Properties of geometric Brownian motion processes

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Geometric Brownian motion as the limit of simpler models.

For large $N$ let $\Delta = \frac{1}{N}$ and every $i\Delta$ units of time a process, $S_\Delta$ increases by $e^{\sigma \sqrt{\Delta}}$ with probability $p = \frac{1}{2} \left(1 + \frac{\mu}{\sigma} \sqrt{\Delta}\right)$ or decreases by $e^{-\sigma \sqrt{\Delta}}$ with probability $1 - p$. Assume $S_\Delta$ only changes value at positive multiples of $\Delta$ and $S_\Delta(0)$.

\[
\begin{align*}
  S_\Delta(i\Delta) &\quad \begin{array}{c} \text{by} \quad e^{\sigma \sqrt{\Delta}} \quad \text{probability} \quad p = \frac{1}{2} \left(1 + \frac{\mu}{\sigma} \sqrt{\Delta}\right) \\
\end{array} \\
  &\quad \text{or} \quad \begin{array}{c}
  \text{by} \quad e^{-\sigma \sqrt{\Delta}} \quad \text{with probability} \quad 1 - p
\end{array} \\
  S_\Delta((i+1)\Delta) &= S_\Delta(i\Delta) \cdot e^{\sigma \sqrt{\Delta}} \\
  S_\Delta((i+1)\Delta) &= S_\Delta(i\Delta) \cdot e^{-\sigma \sqrt{\Delta}}
\end{align*}
\]

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Figure: 3

**Theorem**

As $\Delta \downarrow 0$ one has $S_\Delta \to S$. 
Brownian and Geometric Brownian motion

The Big Picture

\[ S_{\Delta}(0) \xrightarrow{1-p} S_{\Delta}(\Delta) \xrightarrow{p} S_{\Delta}(2\Delta) \xrightarrow{1-p} S_{\Delta}(3\Delta) \]

\[ \Delta = \frac{1}{N} \]

\[ S_{\Delta}(\Delta) \xrightarrow{p} S_{\Delta}(2\Delta) \xrightarrow{1-p} S_{\Delta}(3\Delta) \]

\[ S_{\Delta}(2\Delta) \xrightarrow{1-p} S_{\Delta}(3\Delta) \]

\[ S_{\Delta}(3\Delta) \]

\[ \cdots \]

\[ t = 0 \quad t = \Delta \quad t = 2\Delta \quad t = 3\Delta \quad \cdots \quad t = 1 \]

Figure: 4
Tallying Up

Given a coin that comes up heads with probably $p = \frac{1}{2} \left( 1 + \frac{\mu}{\sigma} \sqrt{\Delta} \right)$ each time it is flipped. Flip the coin $N$ times and let $X$ be the number of heads one obtains. Then the distribution of the values $X$ takes on are described as $\text{BIN}(N, p)$. Let

$$X_j \overset{\text{def}}{=} \text{the number of heads in flip } j \text{ of the coin.}$$

Then $X_1, \cdots, X_N$ are independent and $X \overset{\text{def}}{=} \sum_{j=1}^{N} X_j \sim \text{BIN}(N, p)$. In figure 4, each $\Delta$ interval of time involves a coin flip with a heads increasing $S$ by a factor of $e^{\sigma \sqrt{\Delta}}$ and a tails decreasing $S$ by a factor of $e^{-\sigma \sqrt{\Delta}}$. Thus if $(x_1, x_2, \cdots, x_N)$ consists of 0’s and 1’s and this is the outcome of flipping a coin $N$ times with flip $j$ having $x_j$ heads. The total number of heads is $x = \sum_{j=1}^{N} x_j$. Corresponding to the coin flips is a path through Figure 4 the ends with a stock price at time 1 of

$$\left( e^{\sigma \sqrt{\Delta}} \right)^x \left( e^{-\sigma \sqrt{\Delta}} \right)^{N-x} S(0).$$
To find the break even price of a call with a strike price of $K$:

**Find break even price for $S_\Delta$**

To travel Figure 4 from $t = 0$ to $t = 1$, one is looking at $N$ problems similar to the problem solved in Figure 2. Synthesizing the results of $N$ problems into one result involves careful use of a Theorem called the Arbitrage Theorem (proved with methods from linear programing). Thus one arrives with a strategy for pricing a call option with strike price $K$ for $S_\Delta(1)$ that results in no arbitrage.

**Let $\Delta \downarrow 0$**

Notice that one lets $\Delta \downarrow 0 \iff N \uparrow \infty$. A theorem called the Central Limit Theorem implies $S_\Delta \to S$ and the no–arbitrage strategies for $S_\Delta$ above converge.

\[ \vdots \]

**Black–Scholes Formula**
Black-Scholes Formula

Let

\[ C = \text{the non-arbitrage price for the call option at time } t \]
\[ s = \text{the initial price of the stock, i.e., } S(0) = s \]
\[ K = \text{the strike price for the call option} \]
\[ r = \text{the interest rate per unit time during the exercise time} \]
\[ \sigma^2 = \text{the variance parameter for } S(t), \text{the stock price at time } t \]
\[ \mu = \text{the drift parameter for } S(t) \text{ (not relevant to Black–Scholes Formula)} \]

\[ \Phi(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \, dx = \]

\[ w \overset{\text{def}}{=} \frac{rt + \sigma^2 t/2 - \ln(K/s)}{\sigma \sqrt{t}}. \]

Then

\[ C(s, t, K, \sigma, r) = s\Phi(w) - Ke^{-rt}\Phi(w - \sigma \sqrt{t}). \]
This textbook on the basics of option pricing is accessible to readers with limited mathematical training. It is for both professional traders and undergraduates studying the basics of finance. Assuming no prior knowledge of probability, Sheldon M. Ross offers clear, simple explanations of arbitrage, the Black–Scholes option pricing formula, and other topics such as utility functions, optional portfolio selections, and the capital assets pricing model.
1968 Myron Scholes becomes assistant professor of Finance at MIT and Fisher Black (student of Marvin Minsky) was a consultant for Arthur D. Little in Cambridge.


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The Disaster

1993 John Meriwether starts Long Term Capital Management, a hedge fund in Greenwich, CT. He recruits Robert Merton and Myron Scholes. LTCM has a reputation of having the brightest quants.

1997 Robert Merton and Myron Scholes share the Nobel prize in economics.

1998 LTCM has $3.6 billion in capital - it is the largest fund in existence. In August and September they lose everything and sustain an additional $1.9 billion in personal losses. John Meriwether tries to borrow money from lots of wealthy people, including Warren Buffet and George Soros. They all say no. At the end of September, the Federal Reserve intervenes and takes over LTCM for $3.65 billion.
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### LTCM 1998 Losses

<table>
<thead>
<tr>
<th>Strategy</th>
<th>Loss (in Million)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Russia and other emerging markets</td>
<td>$430 million</td>
</tr>
<tr>
<td>Directional trades in developed countries (shorting Japanese bonds)</td>
<td>$371 million</td>
</tr>
<tr>
<td>Equity pairs trading</td>
<td>$286 million</td>
</tr>
<tr>
<td>Yield-curve arbitrage</td>
<td>$215 million</td>
</tr>
<tr>
<td>Standard &amp; Poor’s 500 stocks</td>
<td>$203 million</td>
</tr>
<tr>
<td>High-yield (junk bond) arbitrage</td>
<td>$100 million</td>
</tr>
<tr>
<td>Interest swaps</td>
<td>$1.6 billion</td>
</tr>
<tr>
<td>Equity volatility bets</td>
<td>$1.3 billion</td>
</tr>
</tbody>
</table>

“The market can stay irrational longer than you can stay solvent.” – John Maynard Keynes
Robert Merton (1944–Present)

Fischer Black (1938–1995)

Myron Scholes (1941–Present)
John Meriwether (1947–Present)
“... In physics, you’re playing against God; in finance, you’re playing against people.” – Emanuel Derman, Goldman, Sachs, and Co.

“There is no subtler, no surer means of overturning the existing basis of society than to debauch the currency. The process engages all the hidden forces of economic law on the side of destruction, and does it in a manner which not one man in a million is able to diagnose.” – John Maynard Keynes

“An economist is someone who sees something happen and wonders whether it would work in theory.” – Ronald Reagan

Stein’s Law: “Things that can’t go on forever, don’t.” – the late economist Herbert Stein, chairman of the Council of Economic Advisers under Presidents Richard Nixon and Gerald Ford
References

www-groups.dcs.st-andrews.ac.uk/~history/Indexes/Full_Chron.html


