Is $1 + 2 + 3 + \ldots = -1/12$?

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Some baby algebra

One of the cornerstones of 8th grade algebra is the associative property of addition:

\[ a + (b + c) = (a + b) + c \]

This can of course be done for any (finite!) list of numbers. You can put the parentheses anywhere you want.

\[ 5 = 1 + (2 + 1) + 1 = (1 + 2) + (1 + 1). \]
A series looks pretty similar. After all, it’s just a list of added numbers...

\[ a_1 + a_2 + a_3 + \ldots \]

Should the associative property apply?
Consider the infinite series

$$1 - 1 + 1 - 1 + \ldots$$

Clearly, by associativity,

$$1 + (-1 + 1) + (-1 + 1) + \ldots = 1$$
Consider the infinite series

\[ 1 - 1 + 1 - 1 + \ldots \]

Clearly, by associativity,

\[ 1 + (\color{red}1 - 1) + (\color{red}1 - 1) + \ldots = 1 \]

But also clearly,

\[ (\color{red}1 - 1) + (\color{red}1 - 1) + \ldots = 0. \]
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What’s the problem?
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So \(1 = 0\)?

What’s the problem?

Divergent series are not associative.
BE CAREFUL WITH INFINITY!
Popular YouTube channel Numberphile (one of the single most popular sources of popular mathematics in the world) ran a video in 2014 that purported to prove the following:

\[ 1 + 2 + 3 + 4 + \ldots = -\frac{1}{12}. \]
The video (6.5 MILLION views!)

ASTOUNDING: $1 + 2 + 3 + 4 + 5 + \ldots = -\frac{1}{12}$

6,540,128 views

Numberphile ©
Published on Jan 9, 2014

Read this too: http://www.bradyparanblog.com/blog/20...
More links & stuff in full description below ↓↓↓

SHOW MORE
It’s in a book, so it must mean SOMETHING...

ASTOUNDING: $1 + 2 + 3 + 4 + 5 + \ldots = -1/12$

6,543,438 views

Numberphile ©
Published on Jan 9, 2014

Read this too: http://www.bradyharanblog.com/blog/20...
More links & staff in full description below!!!
Numberphile’s argument (a game of find the mistakes)

- Define $T_1 = 1 - 1 + 1 - 1 + \ldots$. Say that $T_1 = 1/2$.
- Define $T_2 = 1 - 2 + 3 - 4 + 5\ldots$. Then

\[
2T_2 = (1 - 2 + 3 - 4 + \ldots) + (1 - 2 + 3 - 4 + \ldots)
\]
\[
= 1 + (-2 + 1) + (3 - 2) + (-4 + 3) + \ldots
\]
\[
= 1 - 1 + 1 - 1 = T_1 = 1/2.
\]

and so $T_2 = 1/4$.
- Define $T = 1 + 2 + 3 + \ldots$. Then

\[
T - T_2 = (1 + 2 + 3 + 4 + \ldots) - (1 - 2 + 3 - 4 + \ldots)
\]
\[
= (1 - 1) + (2 - (-2)) + (3 - 3) + (4 - (-4)) + \ldots
\]
\[
= 4 + 8 + 12 + \ldots = 4(1 + 2 + 3 + \ldots) = 4T.
\]

Then since $T - T_2 = 4T$ and $T_2 = 1/4$, $T = -1/12$. 
Numberphile’s argument (a game of find the mistakes)

- Define \( T_1 = 1 - 1 + 1 - 1 + \ldots \) Say that \( T_1 = 1/2 \).

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and so \( T_2 = 1/4 \).

- Define \( T = 1 + 2 + 3 + \ldots \) Then

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and so $T_2 = 1/4$.

- Define $T = 1 + 2 + 3 + \ldots$. Then

$$T - T_2 = (1 + 2 + 3 + 4 + \ldots) - (1 - 2 + 3 - 4 + \ldots)$$

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Then since $T - T_2 = 4T$ and $T_2 = 1/4$, $T = -1/12$. 
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Obviously the **argument** is wrong, but is the **result**? What did they mean?
So is $1 + 2 + 3 + \ldots = -1/12$?

 Obviously the argument is wrong, but is the result? What did they mean? To answer this question, we need to think more deeply than usual about what we mean when we write something like

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots = 1.$$
When we think of convergent series, the intuition is a list of numbers that adds up. But what does that mean?

Let $S_n = \sum_{k=1}^{n} a_k$ be the $n$th partial sum of $\sum a_k$. Then we say that $\sum a_k$ converges to $a$ if $\lim_{n \to \infty} S_n = a$. That is, we assign the number 1 to the series $\sum_{k=1}^{\infty} 2^{-k}$ because $S_n = n \sum_{k=1}^{n} 2^{-k} \to 1$ as $n \to \infty$. More lazily: $1 = \sum_{k=1}^{\infty} 2^{-k}$.
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- More lazily: \( 1 = \sum_{k=1}^{\infty} 2^{-k} \).
A **divergent series** is a series that doesn’t converge. That is, the partial sums do not have a limit. Consider the series

$$1 - 1 + 1 - 1 + 1 \ldots$$

The sequence of partial sums is $1, 0, 1, 0, 1, \ldots$, which doesn’t ever approach a limit, so the series diverges.
Is that really the end of it? Mathematicians like to invent definitions that have meaning. (Like $i^2 = -1$. Why shouldn’t negative numbers have square roots?)

IDEA: On average, the series $1 - 1 + 1 - 1 + \ldots$ is $1/2$. Can this be made meaningful? Is there a "reasonable" way to assign numbers to divergent series (that is, series that can’t be assigned a number by conventional convergence?)

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Cesàro convergence

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1. An infinite series \( \sum a_n \) is called Cesàro summable if the sequence given by the means of the first \( n \) partial sums converge.

2. That is, if \( S_n \) is the \( n \)-th partial sum, let

\[
C_n = \frac{S_1 + S_2 + \ldots + S_n}{n}.
\]

3. If \( \lim_{n \to \infty} C_n \) exists, then \( \sum a_k \) is Cesàro summable, with Cesàro sum \( a \).
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The value of 1 − 1 + 1 − 1 + …

How about 1 − 1 + 1 − 1 + …?

First, compute the partial sums

\[ S_1 = 1, \quad S_2 = 0, \quad S_3 = 1, \quad S_4 = 0, \ldots \]

Now, the \( n \)th means

\[ C_1 = \frac{1}{1}, \quad C_2 = \frac{1 + 0}{2} = \frac{1}{2}, \quad C_3 = \frac{1 + 0 + 1}{3} = \frac{2}{3}, \ldots \]

Then \( C_n = \frac{1}{2} \) for \( n \) even and \( \frac{1 + \frac{n+1}{n}}{n} \) for \( n \) odd.

Then \( \lim_{n \to \infty} C_n = \frac{1}{2} \) and 1 − 1 + 1 − … is summable with sum 1/2.

Even more lazily: 1 − 1 + 1 − 1… “=” \( \frac{1}{2} \).
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3. Then $C_n = \frac{1}{2}$ for $n$ even and $\frac{1}{2} \frac{n+1}{n}$ for $n$ odd.

4. Then $\lim_{n \to \infty} C_n = \frac{1}{2}$ and $1 - 1 + 1 - \ldots$ is summable with sum $1/2$.

5. Even more lazily: $1 - 1 + 1 - 1 \ldots \text{ “=” } \frac{1}{2}$. 
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Cesàro summation is useful

Important properties:

1. Every convergent series is Cesàro summable, and the limits agree. It is an amusing exercise of real analysis to prove this.

2. We can assign numbers to (certain) divergent series in a meaningful way.

3. Cesàro convergent series can be added, and the limit of the sum is the sum of the limits. (Really, we mean that we can treat Cesàro summable series like numbers! We can add series, subtract them, etc.)
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This DOES NOT fix the Numberphile argument! $T_1 = 1 - 1 + 1 - \ldots$ is Cesàro summable but $T_2 = 1 - 2 + 3 - 4 + \ldots$ isn’t!
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We need another idea.
So now what? Let us call on Calculus 2.

Definition
An infinite series is called a $p$-series if $a_n = \frac{1}{n^p}$. That is,

$$\sum a_n = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \ldots$$

A famous example: The harmonic series is the divergent $p$-series with $p = 1$.

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IDEA: AHA!  $T = 1 + 2 + 3 + \ldots$ is a $p$-series with $p = -1$.

$T = 1 + 2 + 3 + \ldots = \frac{1}{1^{-1}} + \frac{1}{2^{-1}} + \frac{1}{3^{-1}} + \ldots$

Unfortunately, since $p \leq 1$, it diverges. Maybe we can figure out a way to assign numbers to $p$-series that diverge.

We should probably start with convergent $p$-series.
A new idea

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The Basel problem

How can we assign numbers to $p$-series? This is an incredibly hard problem.

The Basel problem, first stated in 1644 and not definitively solved for two hundred years, asks the following:

**Basel problem**

What is the exact value of the $p$ series with $p = 2$

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \ldots?$$
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A caveat

I apologize...

I told you to be careful with infinity, and we are about to do some wild (and completely unjustified in 1734) math by pretending that power series are polynomials.

Karl Weierstrass DID fix this... eventually... in 1854 (210 years after the problem was proposed).
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Factoring zeros

**Theorem**

Let \( f \) be a polynomial

\[
f(x) = c_0 + c_1x + \ldots + c_nx^n.
\]

If \( f(a) = 0 \), then \( (1 - \frac{x}{a}) \) is a factor of \( f \).

Usually we see this in the form \((x - a)\), but it should be obvious that \((x - a) = -a(1 - \frac{x}{a})\).

**Example**

\( f(x) = x^2 - 3x + 2 = 0 \) when \( x = 1 \) and \( x = 2 \). So

\[
x^2 - 3x + 2 = 2\left(1 - \frac{x}{2}\right)\left(1 - \frac{x}{1}\right)
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\[
x^2 - 3x + 2 = 2\left(1 - \frac{x}{2}\right)\left(1 - \frac{x}{1}\right)
\]
The Taylor series for $\sin x$ is

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$$

and

$$\frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \ldots$$

So let $f(x) = \frac{\sin x}{x}$ (which we illegally think of as really being an “infinite polynomial”)

A sketch of the solution to the Basel problem
Claim: Since \( f(n\pi) = 0 \) for all \( n \),

\[
\frac{\sin x}{x} = (1 - \frac{x}{\pi})(1 + \frac{x}{\pi})(1 - \frac{x}{2\pi})(1 + \frac{x}{2\pi})\ldots
\]
Claim: Since $f(n\pi) = 0$ for all $n$,

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0 when $x=\pi$ 0 when $x=\pi$
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0 when $x = -\pi$, 
0 when $x = -\pi$
Claim: Since $f(n\pi) = 0$ for all $n$, 

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0 when $x=2\pi$
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0 when $x=2\pi$

Now use the difference of squares formula to get

**Theorem?**

$$\frac{\sin x}{x} = \left(1 - \frac{x^2}{\pi^2}\right)\left(1 - \frac{x^2}{4\pi^2}\right)\left(1 - \frac{x^2}{9\pi^2}\right) \cdots$$
If you liked the last step...

So we have an infinite product of quadratic terms... might as well distribute \( \_\_\_\_\_\_\_\_\_\_\_\_\_\_. \)

\[
\frac{\sin x}{x} = \left(1 - \frac{x^2}{\pi^2}\right) \left(1 - \frac{x^2}{4\pi^2}\right) \left(1 - \frac{x^2}{9\pi^2}\right) \ldots \\
= 1 - \left(\frac{1}{\pi^2} + \frac{1}{4\pi^2} + \frac{1}{9\pi^2} + \ldots\right)x^2 + \ldots
\]

BUT

\[
\frac{\sin x}{x} = 1 - \frac{1}{3!}x^2 + \frac{1}{5!}x^5 - \ldots
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BUT

\[
\frac{\sin x}{x} = 1 - \frac{1}{3!} x^2 + \frac{1}{5!} x^5 - \ldots
\]
A Basel solution

Since the coefficients of (illegally infinite) equal polynomials match, we get

\[
\frac{1}{\pi^2} + \frac{1}{4\pi^2} + \frac{1}{9\pi^2} + \ldots = \frac{1}{3!}
\]

or, after multiplying through by \(\pi^2\) and calculating \(3! = 6\),

\[
\frac{1}{1} + \frac{1}{4} + \frac{1}{9} + \ldots = \frac{\pi^2}{6}.
\]

(Euler verified his solution numerically, since his math wasn’t justified. 110 years later, all of this was shown to work by Weierstrass, but only because \(\sin x\) is very special).
Divergent $p$-series?

Back to our goal: to find the sum of $T = 1 + 2 + 3 + \ldots$. 

Is $1 + 2 + 3 + \ldots = -1/12$?
Divergent $p$-series?

Back to our goal: to find the sum of $T = 1 + 2 + 3 + \ldots$.

We can (with difficulty) assign values to convergent $p$-series.
Divergent $p$-series?

Back to our goal: to find the sum of $T = 1 + 2 + 3 + \ldots$.

We can (with difficulty) assign values to convergent $p$-series. Is there a way to assign values to divergent $p$-series (since Cesàro summation isn’t going to work)?
A functional approach

Definition (p-series version 2: the Zeta function)

Let $\zeta(p) : (1, \infty) \to \mathbb{R}$ be the function

$$\zeta(p) = \sum_{n=1}^{\infty} \frac{1}{n^p}.$$

\(\zeta\) is well-defined, since the output of \(\zeta\) is a convergent \(p\)-series for all \(p \in (1, \infty)\).
A functional approach

Definition (Zeta function)

Let \( \zeta(p) : (1, \infty) \to \mathbb{R} \) be the function

\[
\zeta(p) = \sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{\Gamma(p)} \int_{0}^{\infty} \frac{x^{p-1}}{e^x - 1} \, dx.
\]

where \( \Gamma(p) = \int_{0}^{\infty} x^{p-1} e^x \, dx \).

The function \( \Gamma(p) \) is the function used to recover the factorials: \( \Gamma(n) = (n - 1)! \) when \( n \in \mathbb{N} \). For example, \( \Gamma(5) = 4! \). (It’s cool.)

With this notation, Euler showed

Basel problem - zeta function version

\[
\zeta(2) = \frac{\pi^2}{6}.
\]
A functional approach

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**Basel problem - zeta function version**

$$\zeta(2) = \frac{\pi^2}{6}. $$
Let

\[ \Gamma(z) = \int_0^\infty x^{z-1} e^x \, dx. \]

Exercise 1

1. Use integration by parts to show that \( \Gamma(z + 1) = z\Gamma(z) \).
2. Compute \( \Gamma(1) \).
3. Show that for any positive integer \( n \), \( \Gamma(n + 1) = n! \).
A functional approach

**Definition (Zeta function)**

Let $\zeta(p) : (1, \infty) \rightarrow \mathbb{R}$ be the function

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**Basel problem - zeta function version**

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\zeta(2) = \frac{\pi^2}{6}.
$$
Extension to complex numbers

Another magic step: integral representations can often be immediately extended to complex numbers just by plugging them in!

**Theorem (Chebyshev 1840s)**

\[
\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{x^{s-1}}{e^x - 1} \, dx
\]

is a holomorphic function for \( s \in \mathbb{C} \) with \( \text{Re}(s) > 1 \).

By holomorphic, we mean that \( \zeta \) can be represented by a power series, or equivalently that \( \zeta \) is infinitely differentiable for \( \text{Re}(s) > 1 \).
Bernard Riemann used **analytic continuation** (a powerful technique for extending the domain of holomorphic functions that is sadly beyond the scope of this talk) to show that $\zeta$ satisfies the following functional relation on (almost) the entire complex plane.

**Theorem (Riemann 1859)**

For all $s \in \mathbb{C} \neq 1$, 

$$\zeta(s) = 2^s \pi^{s-1} \sin \left( \frac{\pi s}{2} \right) \Gamma(1 - s) \zeta(1 - s).$$
1. Consider $1 + 2 + 3 + 4 + \ldots$ as a $p$-series with $p = -1$

2. Agree that value of $\zeta(p)$ can be said to represent the “sum” of the $p$-series with exponent $p$ for any value of $p$ (convergent or not).

3. Then

$$1 + 2 + 3 + \ldots = 1 + \frac{1}{1-1} + \frac{1}{2-1} + \ldots$$

$$= \zeta(-1) \text{ (Chebyshev!)}$$

$$= 2^{-1} \pi^{-2} \sin \frac{-\pi}{2} \Gamma(2) \zeta(2) \text{ (Riemann!)}$$

$$= \frac{1}{2\pi^2} (-1)(1!) \frac{\pi^2}{6} \text{ (Euler, Weierstrass, Newton!)}$$

$$= -\frac{1}{12}.$$
1. Consider $1 + 2 + 3 + 4 + \ldots$ as a $p$-series with $p = -1$

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$$1 + 2 + 3 + \ldots = 1 + \frac{1}{1^{1-1}} + \frac{1}{2^{1-1}} + \ldots$$

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Agree that value of $\zeta(p)$ can be said to represent the “sum” of the $p$-series with exponent $p$ for any value of $p$ (convergent or not).

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$$1 + 2 + 3 + \ldots = 1 + \frac{1}{1^{-1}} + \frac{1}{2^{-1}} + \ldots$$

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Yes...
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Yes... IF...
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So is $1 + 2 + 3 + \ldots = -1/12$?

Yes… IF… by “equals” we mean “view $1 + 2 + 3 + \ldots$ as the $p$-series $\frac{1}{1^{-1}} + \frac{1}{2^{-1}} + \frac{1}{3^{-1}} + \ldots$, and then build the zeta function $\zeta(s) = \frac{1}{\Gamma(s)} \int_{0}^{\infty} x^{s-1} e^{-x} dx$ in analogy with the usual convergent $p$-series, and then find the analytic continuation to the whole complex plane, and use THAT to get the relation $\zeta(s) = 2^{s} \pi^{s} \sin \left( \frac{\pi s}{2} \right) \Gamma(1-s) \zeta(1-s)$, and calculate the value of $\zeta(-1)$ and assign it to the series”.

Ryan Tully-Doyle (University of New Haven)
Is $1 + 2 + 3 + \ldots = -1/12$?

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\[ \frac{1}{1^{-1}} + \frac{1}{2^{-1}} + \frac{1}{3^{-1}} + \ldots, \]

and then build the zeta function

\[ \zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{x^{s-1}}{e^x - 1} \, dx \]

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Weasel words

ASTOUNDING: $1 + 2 + 3 + 4 + 5 + \ldots = -\frac{1}{12}$

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Ryan Tully-Doyle (University of New Haven) Is $1 + 2 + 3 + \ldots = -1/12$?
into why Lorentz invariance would be lost in the light-cone approach for the wrong $A$ or $D$.

First, we assert that the operator ordering constant in the Hamiltonian for a free field comes from summing the zero-point energies of each oscillator mode, $\frac{1}{2}\omega$ for a bosonic field like $X^a$. Equivalently, it always works out that the natural operator order is averaged, $\frac{1}{2}\omega (aa^\dagger + a^\dagger a)$, which is the same as $\omega (a^\dagger a + \frac{1}{2})$. In $H$ this would give

$$A = \frac{D - 2}{2} \sum_{n=1}^{\infty} n,$$  \hspace{1cm} (1.3.31)

the factor of $D - 2$ coming from the sum over transverse directions. The zero-point sum diverges. It can be evaluated by regulating the theory and then being careful to preserve Lorentz invariance in the renormalization. This leads to the odd result

$$\sum_{n=1}^{\infty} n \to -\frac{1}{12}.$$  \hspace{1cm} (1.3.32)

To motivate this, insert a smooth cutoff factor

$$\exp(-e\gamma^{-1/2}_\sigma |k_\sigma|)$$  \hspace{1cm} (1.3.33)

into the sum, where $k_\sigma = n\pi/\ell$ and the factor of $\gamma^{-1/2}_\sigma$ is included to make this invariant under $\sigma$ reparameterizations. The zero-point constant is then

$$A \to \frac{D - 2}{2} \sum_{n=1}^{\infty} n \exp\left[-e\pi/2p^+\gamma' \ell \right]$$

$$= \frac{D - 2}{2} \left(\frac{2\ell p^+ \gamma'}{e^2\pi} - \frac{1}{12} + O(e)\right).$$  \hspace{1cm} (1.3.34)

The cutoff-dependent first term is proportional to the length $\ell$ of the string and can be canceled by a counterterm in the action proportional to $(\ell^2 p^+ \gamma')^{1/2}$. In fact, Weyl invariance requires that it be canceled, leaving...
I leave you with two quotes:

- “In mathematics, you don’t understand things. You just get used to them.” – J. von Neumann (allegedly said of the proof of the existence of the analytic continuation of $\zeta$).

- “Divergent series are an invention of the devil, and it is a disgrace to base any proof on them.” – N. H. Abel (who famously invented a way to sum them).
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