INSTITUTE OF PHYSICS PUBLISHING CLASSICAL AND QUANTUM GRAVITY

Class. Quantum Grav. **23** (2006) 2011–2020 doi:10.1088/0264-9381/23/6/011

Acceleration of the universe in the Einstein frame of a metric-affine $f(R)$ gravity

Nikodem J Popławski

Department of Physics, Indiana University, 727 East Third Street, Swain Hall West 117, Bloomington, IN 47405-7105, USA

E-mail: nipoplaw@indiana.edu

Received 8 October 2005, in final form 23 January 2006
Published 28 February 2006
Online at stacks.iop.org/CQG/23/2011

Abstract

We show that inflation and current cosmic acceleration can be generated by a metric-affine $f(R)$ gravity formulated in the Einstein conformal frame, if the gravitational Lagrangian $L(R)$ contains both positive and negative powers of the curvature scalar R. In this frame, we give the equations for the expansion of the homogeneous and isotropic matter-dominated universe in the case $L(R) = R + \frac{R^3}{\beta^2} - \frac{\alpha^2}{3R}$, where α and β are constants. We also show that gravitational effects of matter in such a universe at very late stages of its expansion are weakened by a factor that tends to 3/4, and the energy density of matter ϵ scales the same way as in the ΛCDM model only when $\kappa\epsilon \ll \alpha$.

PACS numbers: 04.50.+h, 98.80.−k

1. Introduction

Recent observations of type Ia supernovae [1] and the cosmic microwave background radiation [2, 3] indicate that the universe is undergoing a phase of accelerated expansion. It is also believed that shortly after the big bang, the universe passed through a phase of extremely rapid expansion (inflation) [4]. Such an idea resolves several problems in big bang cosmology, e.g., the flatness problem and the horizon problem. The most accepted explanation of current cosmic acceleration is that the universe is dominated by dark energy [5], whereas inflation is thought to be driven by scalar fields with an appropriate potential.

However, it is also possible to modify Einstein's general relativity to obtain gravitational field equations that allow both inflation and present cosmic acceleration. A particular class of alternative theories of gravity that has recently attracted a lot of interest is that of the $f(R)$ gravity models, in which the gravitational Lagrangian is a function of the curvature scalar R [6]. It has been shown that current acceleration may originate from the addition of a term R^{-1} (or other negative powers of R) to the Einstein–Hilbert Lagrangian R, whereas terms with positive powers of R may be the source of inflation [7, 8]. As in general relativity, these models obtain the field equations by varying the total action for both the field and matter.

There are two different approaches for how to vary the action in these models: metric and metric-affine. The first one is the usual Einstein–Hilbert variational principle, according to which the action is varied with respect to the metric tensor $g_{\mu\nu}$, and the affine connection is given by the Christoffel symbols (the Levi-Civita connection) [9]. The other one is the Palatini variational principle (originally formulated by Einstein), according to which the metric and the connection are considered as geometrically independent quantities, and the action must be varied with respect to both of them [10]. Both the metric and the metric-affine approaches give the same result only if we use the standard Einstein–Hilbert action, since variation with respect to the connection gives the usual expression for the Christoffel symbols.

Of the two approaches, the metric-affine formalism seems more general since it requires one less constraint than the metric approach (no *a priori* relation between the metric and the connection). Moreover, the field equations in this formalism are second-order differential equations (and the Cauchy problem is similar to that in general relativity), whereas in metric theories they are fourth-order [11]. Another remarkable property of the metric-affine approach is that the field equations in vacuum reduce to the standard Einstein equations of general relativity with a cosmological constant [12], whereas vacuum metric theories allow both accelerated and decelerated phases of the universe expansion [13]. This approach is also free of instabilities that appear in the metric formulation of $1/R$ gravity [14], although such instabilities can be suppressed by adding to the Lagrangian terms with positive powers of R [15], or by quantum effects [16]. Furthermore, there is a debate on the compatibility of $f(R)$ gravities with solar system observations [17], and on the Newtonian limit of these theories [18]. There is theoretical evidence suggesting that metric-affine models with the inverse power of the curvature scalar have a good Newtonian limit [19], and that metric models pass the solar system tests [15, 20].

One can show that any of these theories of gravitation is conformally equivalent to the Einstein theory of the gravitational field interacting with additional matter fields [21]. However, a GR-like formulation can be obtained without any redefinition of the metric, by isolating the spin-0 degree of freedom due to the occurrence of nonlinear second-order terms in the Lagrangian, and encoding it into an auxiliary scalar field ϕ by means of a Legendre transformation. Such a transformation in classical mechanics replaces the Lagrangian of a mechanical system with the Helmholtz Lagrangian [22].

The set of variables $(g_{\mu\nu}, \phi)$ is commonly called the *Jordan conformal frame*, although it refers to a choice of dynamical variables rather than to a choice of a frame of reference. In the Jordan frame of metric $f(R)$ theories, the self-gravitating scalar field ϕ violates the stability of vacuum and the positivity of energy [11, 22]. These unphysical properties can be eliminated by a certain conformal transformation of the metric: $g_{\mu\nu} \rightarrow h_{\mu\nu} = f'(R)g_{\mu\nu}$. The new set $(h_{\mu\nu}, \phi)$ is called the *Einstein conformal frame*. Although both frames are equivalent mathematically, they are *not* equivalent physically [23], and the interpretation of cosmological observations can drastically change depending on the adopted frame [24]. The physically measured metric is determined from the coupling to matter, and the principle of equivalence can provide an operational definition of the metric tensor [25]. Furthermore, in the Einstein frame, the principle of equivalence is violated [23] and the available tests of this principle can serve as the constraints on nonlinear gravities.

In the metric-affine formalism, the auxiliary field ϕ has no kinetic term and does not violate the positivity of energy. Therefore, which frame is physical is a matter of choice, although this question should be ultimately answered by experiment or observation. Remarkably, it is the Einstein frame in which the connection is metric-compatible. Therefore, in this work we treat $h_{\mu\nu}$ as the physical metric tensor [11, 22, 23].

The authors of [7, 8] applied the metric variational formalism to $f(R)$ theories of gravitation. It has been shown that positive and negative powers of the curvature scalar in the gravitational Lagrangian can cause inflation and current acceleration also in the metric-affine formalism [26–29], although the compatibility of Palatini $f(R)$ models with experiment is being debated [30, 31], and these models may face the problem of stability of matter perturbations [32]. The simplest Lagrangian in a metric-affine theory, which drives both phases of acceleration, has the form $L(R) = R + \frac{R^3}{\beta^2} - \frac{\alpha^2}{3R}$ [33]. Here, α and β are constants, and a cubic term was chosen because a quadratic term R^2 cannot lead to gravity driven inflation in the Palatini formalism [34]. We emphasize that, in both formalisms, the above authors studied cosmology in the *Jordan frame*.

In this paper, we show that inflation and present cosmic acceleration can be generated by a metric-affine $f(R)$ gravity formulated in the *Einstein frame*, if the Lagrangian contains both positive and negative powers of the curvature scalar. In this frame, we give explicit formulae for the expansion of the homogeneous and isotropic matter-dominated universe, using the Lagrangian of [33]. We also show that gravitational effects of matter in such a universe at late stages of its expansion are weakened, and the energy density of matter differs in scaling from that in the ΛCDM model.

In section 2, we introduce the metric-affine formalism for an $f(R)$ gravity in the Einstein frame. In section 3, we apply the gravitational field equations to a homogeneous and isotropic universe, and study them for the above Lagrangian. The results are summarized in section 4.

2. Metric-affine formalism in the Einstein conformal frame

In this section we review the metric-affine variational approach to a gravitational theory [35]. The equations of the field are obtained from the Palatini variational principle, according to which both the metric tensor $g_{\mu\nu}$ and the affine connection $\Gamma^{\rho}_{\mu\nu}$ are regarded as independent variables. The action for an $f(R)$ gravity in the original (Jordan) frame is given by

$$S_J = -\frac{1}{2\kappa c}\int d^4x[\sqrt{-\tilde{g}}L(\tilde{R})] + S_m(\tilde{g}_{\mu\nu}, \psi). \tag{1}$$

Here, $\sqrt{-\tilde{g}}L(\tilde{R})$ is a Lagrangian density that depends on the curvature scalar $\tilde{R}$, S_m is the action for matter represented symbolically by ψ, and $\kappa = \frac{8\pi G}{c^4}$. Tildes indicate quantities calculated in the Jordan frame. The curvature scalar is obtained by contracting the Ricci tensor $R_{\mu\nu}$ with the metric tensor,

$$\tilde{R} = R_{\mu\nu}(\Gamma)\tilde{g}^{\mu\nu}, \tag{2}$$

and the Ricci tensor depends on the symmetric connection $\Gamma^{\rho}_{\mu\nu}$,

$$R_{\mu\nu}(\Gamma) = \partial_\rho\Gamma^{\rho}_{\nu\mu} - \partial_\nu\Gamma^{\rho}_{\rho\mu} + \Gamma^{\sigma}_{\nu\mu}\Gamma^{\rho}_{\rho\sigma} - \Gamma^{\sigma}_{\rho\mu}\Gamma^{\rho}_{\nu\sigma}. \tag{3}$$

Variation of the action with respect to $g_{\mu\nu}$ yields the field equations,

$$L'(\tilde{R})R_{\mu\nu} - \tfrac{1}{2}L(\tilde{R})\tilde{g}_{\mu\nu} = \kappa T_{\mu\nu}, \tag{4}$$

where the dynamical energy–momentum tensor of matter is generated by the Jordan metric tensor,

$$\delta S_m = \frac{1}{2c}\int d^4x\sqrt{-\tilde{g}}T_{\mu\nu}\delta\tilde{g}^{\mu\nu}, \tag{5}$$

and the prime denotes the derivative of a function with respect to its argument. If we assume that S_m is independent of $\Gamma^{\rho}_{\mu\nu}$, then variation of the action with respect to the connection leads to

$$\nabla_{\rho}(L'(\tilde{R})\tilde{g}^{\mu\nu}\sqrt{-\tilde{g}}) = 0, \tag{6}$$

from which it follows that the affine connection coefficients are the Christoffel symbols,

$$\Gamma^{\rho}_{\mu\nu} = \left\{ {\rho \atop \mu\nu} \right\}_g = \tfrac{1}{2}g^{\rho\lambda}(\partial_{\mu}g_{\nu\lambda} + \partial_{\nu}g_{\mu\lambda} - \partial_{\lambda}g_{\mu\nu}), \tag{7}$$

with respect to the conformally transformed metric

$$g_{\mu\nu} = L'(\tilde{R})\tilde{g}_{\mu\nu}. \tag{8}$$

The metric $g_{\mu\nu}$ (denoted by $h_{\mu\nu}$ in section 1) defines the Einstein frame with a geodesic structure (metric-compatible connection).

One can show that the action (1) is dynamically equivalent to the following Helmholtz action [11, 22]:

$$S_H = -\frac{1}{2\kappa c}\int \mathrm{d}^4x\sqrt{-\tilde{g}}[L(\phi(p)) + p(\tilde{R} - \phi(p))] + S_m(\tilde{g}_{\mu\nu}, \psi), \tag{9}$$

where p is a new scalar variable. The function $\phi(p)$ is determined by

$$\left.\frac{\partial L(\tilde{R})}{\partial \tilde{R}}\right|_{\tilde{R}=\phi(p)} = p. \tag{10}$$

From equations (8) and (10) it follows that

$$\phi = RL'(\phi). \tag{11}$$

In the Einstein frame, the action (9) becomes the standard Einstein–Hilbert action of general relativity with additional non-kinetic scalar field terms:

$$S_E = -\frac{1}{2\kappa c}\int \mathrm{d}^4x\sqrt{-g}\left[R - \frac{\phi(p)}{p} + \frac{L(\phi(p))}{p^2}\right] + S_m(p^{-1}g_{\mu\nu}, \psi). \tag{12}$$

Here, R is the curvature scalar of the metric $g_{\mu\nu}$. Choosing ϕ as the variable leads to

$$S_E = -\frac{1}{2\kappa c}\int \mathrm{d}^4x\sqrt{-g}[R - 2V(\phi)] + S_m([L'(\phi)]^{-1}g_{\mu\nu}, \psi), \tag{13}$$

where $V(\phi)$ is the effective potential,

$$V(\phi) = \frac{\phi L'(\phi) - L(\phi)}{2[L'(\phi)]^2}. \tag{14}$$

Variation of the action (13) with respect to $g_{\mu\nu}$ yields

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \frac{\kappa T_{\mu\nu}}{L'(\phi)} - V(\phi)g_{\mu\nu}, \tag{15}$$

and variation with respect to ϕ gives

$$-2V'(\phi) = \kappa T\frac{L''(\phi)}{[L'(\phi)]^2}, \tag{16}$$

where $T = T_{\mu\nu}g^{\mu\nu}$. Equations (15) and (16) are the equations of the gravitational field in the Einstein frame [30, 35]. One can easily check that equation (16) is not independent, it results from equations (11), (14) and (15). From equation (15) it follows that the tensor $T_{\mu\nu}$ is not covariantly conserved, unlike that in the Jordan frame [36][1].

Contracting equation (15) with the metric tensor $g^{\mu\nu}$ gives

$$R = -\frac{\kappa T}{L'(\phi)} + 4V(\phi), \tag{17}$$

which is an equation for R since both ϕ and T depend only on R due to equations (11) and (16). This is equivalent to

$$\phi L'(\phi) - 2L(\phi) = \kappa T L'(\phi), \tag{18}$$

which is an equation for ϕ as a function of T, and called the structural equation [27]. For the case of $T = 0$ which holds at the early stages of the universe (relativistic matter) and, to good approximation, during advanced cosmic acceleration (when $\kappa T \ll |\phi|$), we obtain

$$\phi L'(\phi) - 2L(\phi) = 0, \tag{19}$$

which agrees with the structural equation in vacuum for an $f(R)$ gravity in the Jordan frame [26]. Equation (19) gives $\phi = \text{const}$, which, upon substitution into equation (15), leads to the Einstein equations of general relativity with a cosmological constant [12] and the gravitational coupling κ modified by a constant factor $L'(\phi)$. Therefore, inflation and current cosmic acceleration can be generated by a metric-affine $f(R)$ gravity formulated in the Einstein frame if the gravitational Lagrangian $L(R)$ contains both positive and negative powers of the curvature scalar, because such a possibility results from the structural equation.

Let us consider the Lagrangian [33],

$$L(\phi) = \phi + \frac{\phi^3}{\beta^2} - \frac{\alpha^2}{3\phi}, \tag{20}$$

where β and α are positive constants, remembering that ϕ is the curvature scalar in the Jordan frame, $\tilde{R}$. Equation (19) reads

$$\phi^4 - \beta^2\phi^2 + \alpha^2\beta^2 = 0, \tag{21}$$

and for $\alpha \ll \beta$ it has two de Sitter solutions:

$$\phi_{\text{inf}} = -\beta, \qquad \phi_{\text{ca}} = -\alpha, \tag{22}$$

describing inflation and present acceleration, respectively [8, 33]. The corresponding values of the curvature scalar in the Einstein frame are obtained with equation (11):

$$R_{\text{inf}} = -\frac{\beta}{4}, \qquad R_{\text{ca}} = -\frac{3\alpha}{4}. \tag{23}$$

If $T \neq 0$, equation (18) leads to a quintic equation for ϕ as a function of T.

[1] By covariant conservation of a tensor, we mean that the divergence of this tensor vanishes using $\Gamma^{\rho}_{\mu\nu}$ in the definition of the covariant derivative. If instead of $\Gamma^{\rho}_{\mu\nu}$, we used Christoffel symbols associated with $\tilde{g}_{\mu\nu}$ in this definition, then the energy–momentum tensor from equation (5), which is associated with $\tilde{g}_{\mu\nu}$ as well, would be automatically conserved [9, 36].

Instead of $T_{\mu\nu}$, one might have interpreted the energy–momentum tensor generated by the geodesic metric $g_{\mu\nu}$ as the true energy–momentum tensor of matter. Such a tensor, equal to the first term on the right-hand side of equation (15), is covariantly conserved according to the above definition, since the connection $\Gamma^{\rho}_{\mu\nu}$ is compatible with $g_{\mu\nu}$. Yet this equation would imply the constancy of $V(\phi)$, which, with equation (14), yields $L(\phi) = \phi$. Therefore, we would arrive at general relativity with a cosmological constant, which is not interesting from a modified gravity perspective [35].

3. Inflation and current cosmic acceleration in FLRW cosmology

We now proceed to study a Friedmann–Lemaître–Robertson–Walker (FLRW) cosmology driven by the above field equations. Let us consider a homogeneous and isotropic universe which is spatially flat [3]. In this case, the interval is given by

$$ds^2 = c^2\,dt^2 - a^2(t)(dx^2 + dy^2 + dz^2), \tag{24}$$

where $a(t)$ is the scale factor. Moreover, the energy–momentum tensor of matter in the co-moving frame of reference is of the form

$$T_\mu^\nu = \mathrm{diag}(\epsilon, -P, -P, -P), \tag{25}$$

where ϵ is the energy density and P denotes the pressure. Clearly, $T = \epsilon - 3P$. In the matter-dominated universe we can approximate matter by dust and set $P = 0$.

The Hubble parameter $H = \frac{\dot{a}}{a}$ can be obtained from the 00 component of equation (15), which for the metric (24) and for dust becomes

$$H(\phi) = c\sqrt{\frac{\phi L'(\phi) - 3L(\phi)}{6[L'(\phi)]^2}}. \tag{26}$$

The dot denotes the time-derivative and we used equation (18). At very late stages of the universe expansion ($T \sim 0$) equations (18) and (26) give

$$H(\phi) = c\sqrt{-\frac{L(\phi)}{6[L'(\phi)]^2}}. \tag{27}$$

For early stages we must use the ultrarelativistic equation of state, $P = \frac{\epsilon}{3}$, for which T vanishes as well. In this case, the Hubble parameter is given by

$$H(\phi) = c\sqrt{\frac{L(\phi) - \phi L'(\phi)}{6[L'(\phi)]^2}}. \tag{28}$$

If we use the Lagrangian (20), the Einstein frame values of H for the two de Sitter phases become

$$H_{\mathrm{inf}} = c\sqrt{\beta/48}, \qquad H_{\mathrm{ca}} = c\sqrt{\alpha/16}. \tag{29}$$

For lower temperatures, matter becomes non-relativistic and T increases. Since T deviates from zero, ϕ ceases to be constant and the field equations differ from those with a cosmological constant. The universe gradually becomes matter-dominated, and undergoes a transition from inflationary acceleration to a decelerated expansion phase. At some moment, T reaches a maximum and then decreases. When κT becomes much smaller than $|\phi|$, the universe undergoes a smooth transition back to an exponential acceleration.

We do not establish the kind of matter considered here since, for cosmological purposes, matter in a homogeneous and isotropic universe can be simply described by the energy density and pressure related to each other by the effective equation of state. In the late universe, the kind of matter does not influence the deceleration–acceleration transition because matter is non-relativistic ($P = 0$). For the early universe, the composition of matter is crucial to establish when and how the transition from inflation to the matter-dominated epoch occurs. Since the purpose of this paper is to show that inflation and present cosmic acceleration *can* arise from nonlinear gravity in the Einstein frame, we did not address the particle physics of the inflation–deceleration transition.

For dust, the covariant conservation of the right-hand side of equation (15) leads to

$$L'(\phi)\frac{d}{dt}\left[\frac{\epsilon(\phi)}{L'(\phi)} - \frac{V(\phi)}{\kappa}\right] + 3H(\phi)\epsilon(\phi) = 0, \tag{30}$$

which with equations (14) and (18) reads

$$\dot{\phi} = \frac{6L'H(\phi L' - 2L)}{2L'^2 + \phi L'L'' - 6LL''}. \tag{31}$$

From now on, the prime denotes the ϕ-derivative. Therefore, making use of equation (26) we obtain

$$\dot{\phi} = \frac{\sqrt{6}c(\phi L' - 2L)\sqrt{\phi L' - 3L}}{2L'^2 + \phi L'L'' - 6LL''}, \tag{32}$$

which can be integrated for a given function $L = L(\phi)$, yielding $\phi = \phi(t)$ and $R = R(t)$. Combining equations (26) and (32) gives the function $H = H(t)$, from which we obtain the final expression for expansion, $a = a(t)$.

For the Lagrangian (20), equation (26) reads

$$\frac{H^2}{c^2} = \frac{-\frac{\phi}{3} + \frac{2\alpha^2}{9\phi}}{\left(1 + \frac{3\phi^2}{\beta^2} + \frac{\alpha^2}{3\phi^2}\right)^2}, \tag{33}$$

and equation (32) becomes

$$\dot{\phi} = \frac{c\left(-\phi + \frac{\phi^3}{\beta^2} + \frac{\alpha^2}{\phi}\right)\sqrt{\frac{2\alpha^2}{\phi} - 3\phi}}{1 - \frac{2\alpha^4}{3\phi^4} - \frac{9\phi^2}{\beta^2} + \frac{7\alpha^2}{3\phi^2} + \frac{10\alpha^2}{\beta^2}}. \tag{34}$$

In the Jordan frame, the equation for $\dot{\phi}$ is more complicated (see [33]). From the time-dependence of ϕ we can derive the time-dependence of R by simply applying equation (11). We give the approximate expressions for $\dot{\phi}$ for three regions: inflation ($\phi \sim -\beta$), the matter-dominated era ($-\alpha \gg \phi \gg -\beta$) and advanced cosmic acceleration ($\phi \sim -\alpha$). In the course of expansion, the quantity ϕ varies between $-\beta$ and $-\alpha$ (the two de Sitter values).

In the first case, we use $\phi \ll -\alpha$ to obtain

$$\frac{H^2}{c^2} = -\frac{\phi}{3\left(1 + \frac{3\phi^2}{\beta^2}\right)^2} \tag{35}$$

and

$$\dot{\phi} = \frac{c\left(-\phi + \frac{\phi^3}{\beta^2}\right)\sqrt{-3\phi}}{1 - \frac{9\phi^2}{\beta^2}}. \tag{36}$$

We see that this expression becomes singular at $\phi = -\frac{\beta}{3}$. We note, however, that in our derivation of equation (32) we assumed $P = 0$. Therefore, the condition $\phi = -\frac{\beta}{3}$ is the limit of validity of this assumption, i.e., may be regarded as the limit of non-relativisticity of matter. In order to derive the universe expansion for smaller values of ϕ (down to $-\beta$), we must use a relativistic equation of state, for which $P \neq 0$.

In the second case, we neglect all terms with α or β, arriving at

$$\frac{H^2}{c^2} = -\frac{\phi}{3}, \qquad \dot{\phi} = \sqrt{3}c(-\phi)^{3/2}. \tag{37}$$

Since now $L'(\phi) \approx 1$ and $\phi \approx R$, the above formulae reproduce the standard Friedmann cosmology: $R \propto \epsilon \propto t^{-2}, a \propto t^{2/3}$. Therefore, this region of time also corresponds to the radiation-dominated era and big bang nucleosynthesis. Note that the Friedmann equations may be obtained as a limiting case only if $\alpha \ll \beta$.

In the third case, we have

$$\frac{H^2}{c^2} = \frac{-\frac{\phi}{3} + \frac{2\alpha^2}{9\phi}}{\left(1 + \frac{\alpha^2}{3\phi^2}\right)^2}, \tag{38}$$

and equation (32) gives

$$\dot{\phi} = \frac{c\left(-\phi + \frac{\alpha^2}{\phi}\right)\sqrt{\frac{2\alpha^2}{\phi} - 3\phi}}{1 - \frac{2\alpha^4}{3\phi^4} + \frac{7\alpha^2}{3\phi^2}}. \tag{39}$$

In the course of time $\phi \to -\alpha$ and $\dot{\phi} \to 0$, and the universe asymptotically approaches a de Sitter expansion [26].

Finally, we derive the Einstein equations in the universe at very late stages of its expansion (when $\kappa T \ll \alpha$) in the linear and quadratic approximation of a small quantity $\frac{\kappa T}{\alpha}$. Equation (18) becomes cubic in ϕ,

$$\phi^3 + \kappa T \phi^2 - \alpha^2 \phi + \frac{\alpha^2 \kappa T}{3}, \tag{40}$$

and its solution, which deviates from $-\alpha$ by a term linear in $\frac{\kappa T}{\alpha}$, is

$$\phi = -\alpha - \tfrac{2}{3}\kappa T + O(T^2). \tag{41}$$

In this approximation, we obtain

$$L(\phi) = -\frac{2\alpha}{3}\left(1 + \frac{4\kappa T}{3\alpha}\right), \qquad L'(\phi) = \frac{4}{3}\left(1 - \frac{\kappa T}{6\alpha}\right), \tag{42}$$

and the Einstein equations become

$$R_{\mu\nu} - \tfrac{1}{2} R g_{\mu\nu} = \tfrac{3}{4}\kappa T_{\mu\nu} + \Lambda g_{\mu\nu}, \tag{43}$$

where

$$\Lambda = \frac{3\alpha}{16} \tag{44}$$

plays the role of a cosmological constant. We see that the coupling between matter and the gravitational field is decreased by a factor of 3/4, as in the Jordan frame [29]. The conservation law for the energy density (30) in this limit becomes

$$\dot{\epsilon} + 3H\epsilon = 0, \tag{45}$$

which gives the usual scaling of non-relativistic matter, $\epsilon \propto a^{-3}$, as in the ΛCDM model.

The solution of equation (18), which deviates from $-\alpha$ by terms linear and quadratic in $\frac{\kappa T}{\alpha}$, appears to be the same as that in the linear approximation:

$$\phi = -\alpha - \tfrac{2}{3}\kappa T + O(T^3). \tag{46}$$

Consequently, we obtain

$$\begin{aligned} L'(\phi) &= \frac{4}{3}\left(1 - \frac{\kappa T}{3\alpha} + \frac{\kappa^2 T^2}{3\alpha^2}\right), \\ V(\phi) &= -\frac{3\alpha}{16} + \frac{\kappa^2 T^2}{16\alpha}, \end{aligned} \tag{47}$$

and the Einstein equations become

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \frac{3}{4}\kappa T_{\mu\nu}\left(1 + \frac{\kappa T}{3\alpha}\right) + \Lambda g_{\mu\nu}, \tag{48}$$

where the cosmological constant is now a function of T,

$$\Lambda = \frac{3\alpha}{16} - \frac{\kappa^2 T^2}{16\alpha}. \tag{49}$$

The coupling between matter and the gravtitational field is decreased by a factor which tends to $3/4$ as $T \to 0$. The conservation law for the energy density (30) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\epsilon + \frac{\kappa\epsilon^2}{4\alpha}\right) + H\left(3\epsilon + \frac{\kappa\epsilon^2}{\alpha}\right) = 0, \tag{50}$$

which gives the energy density scaling that differs from the ΛCDM scaling, and tends to it as $\epsilon \to 0$ ($\kappa\epsilon \ll \alpha$).

4. Summary

Inflation and current cosmic acceleration can be generated by replacing the GR Einstein–Hilbert Lagrangian with a modified $f(R)$ Lagrangian, which offers an alternative explanation of cosmological acceleration. We used the metric-affine variational formalism, and chose the Einstein conformal frame as being physical. In this frame, we derived explicit formulae for the matter-dominated universe expansion for the particular case $L(R) = R + \frac{R^3}{\beta^2} - \frac{\alpha^2}{3R}$, and showed that they reproduce the standard Friedmann cosmology in the region of middle values of R. We also demonstrated that gravitational effects of matter in such a universe at very late stages of its expansion are weakened by a factor that tends to the Jordan frame value $3/4$. Finally, we showed that the energy density of matter in the late universe scales according to the same power law as in the ΛCDM model only in the limit $\epsilon \to 0$.

We did not address the physical scenario of the inflation–deceleration transition. We also did not address the problem of constraining the values of the constants α and β from astronomical observations. Such studies, as well as determination of the function $f(R)$ from the data, as has been explored in the metric approach [37], will be the subject of future work.

References

[1] Riess A G *et al* 1998 *Astron. J.* **116** 1009
Perlmutter S *et al* 1999 *Astrophys. J.* **517** 565
[2] Halverson N W *et al* 2002 *Astrophys. J.* **568** 38
[3] Netterfield C B *et al* 2002 *Astrophys. J.* **571** 604
Bennett C L *et al* 2003 *Astrophys. J. Suppl. Ser.* **148** 1
[4] Guth A H 1981 *Phys. Rev.* D **23** 347
[5] Peebles P J E and Ratra B 2003 *Rev. Mod. Phys.* **75** 559
Padmanabhan T 2003 *Phys. Rep.* **380** 235
[6] Barrow J D and Ottewill A C 1983 *J. Phys. A: Math. Gen.* **16** 2757
[7] Starobinsky A A 1980 *Phys. Lett.* B **91** 99
Capozziello S, Carloni S and Troisi A 2003 *Recent Research Developments in Astronomy and Astrophysics* RSP/AA/21-2003 (*Preprint* astro-ph/0303041)
Carroll S M, Duvvuri V, Trodden M and Turner M 2004 *Phys. Rev.* D **70** 043528
[8] Nojiri S and Odintsov S D 2003 *Phys. Rev.* D **68** 123512
[9] Landau L D and Lifshitz E M 1975 *The Classical Theory of Fields* (Oxford: Pergamon)
[10] Einstein A 1925 *Sitz.-ber. Preuss. Akad. Wiss.* 414
Magnano G, Ferraris M and Francaviglia M 1987 *Gen. Rel. Grav.* **19** 465
[11] Magnano G 1995 *Italian General Relativity* pp 213–34 (*Preprint* gr-qc/9511027)
[12] Ferraris M, Francaviglia M and Volovich I 1993 *Preprint* gr-qc/9303007
Hamity V H and Barraco D E 1993 *Gen. Rel. Grav.* **25** 461
Ferraris M, Francaviglia M and Volovich I 1994 *Class. Quantum Grav.* **11** 1505

[13] Das S, Banerjee N and Dadhich N 2005 *Preprint* astro-ph/0505096
[14] Dolgov A D and Kawasaki M 2003 *Phys. Lett.* B **573** 1
Meng X H and Wang P 2003 *Class. Quantum Grav.* **20** 4949
Soussa M E and Woodard R P 2004 *Gen. Rel. Grav.* **36** 855
[15] Nojiri S and Odintsov S D 2004 *Gen. Rel. Grav.* **36** 1765
[16] Nojiri S and Odintsov S D 2004 *Mod. Phys. Lett.* A **19** 627
[17] Chiba T 2003 *Phys. Lett.* B **575** 1
[18] Dick R 2004 *Gen. Rel. Grav.* **36** 217
Meng X H and Wang P 2004 *Gen. Rel. Grav.* **36** 1947
Dominguez A E and Barraco D E 2004 *Phys. Rev.* D **70** 043505
Olmo G J 2005 *Phys. Rev. Lett.* **95** 261102 (*Preprint* gr-qc/0505101)
[19] Sotiriou T P 2005 *Preprint* gr-qc/0507027
[20] Cembranos J A R 2005 *Preprint* gr-qc/0507039
[21] Whitt B 1984 *Phys. Lett.* B **145** 176
Jakubiec A and Kijowski J 1988 *Phys. Rev.* D **37** 1406
Barrow J D and Cotsakis S 1988 *Phys. Lett.* B **214** 515
Maeda K 1989 *Phys. Rev.* D **39** 3159
Wands D 1994 *Class. Quantum Grav.* **11** 269
[22] Magnano G and Sokolowski L M 1994 *Phys. Rev.* D **50** 5039
[23] Faraoni V and Gunzig E 1999 *Int. J. Theor. Phys.* **38** 217
[24] Allemandi G, Capone M, Capozziello S and Francaviglia M 2006 *Gen. Rel. Grav.* **38** 33 (*Preprint* hep-th/0409198)
[25] Brans C H 1988 *Class. Quantum Grav.* **5** L197
[26] Vollick D N 2003 *Phys. Rev.* D **68** 063510
[27] Allemandi G, Borowiec A and Francaviglia M 2004 *Phys. Rev.* D **70** 043524
Allemandi G, Borowiec A, Francaviglia M and Odintsov S D 2005 *Phys. Rev.* D **72** 063505
[28] Meng X H and Wang P 2004 *Class. Quantum Grav.* **21** 951
[29] Kremer G M and Alves D S M 2004 *Phys. Rev.* D **70** 023503
[30] Flanagan E E 2004 *Phys. Rev. Lett.* **92** 071101
[31] Vollick D N 2004 *Class. Quantum Grav.* **21** 3813
Flanagan E E 2004 *Class. Quantum Grav.* **21** 417
[32] Koivisto T and Kurki-Suonio H 2005 *Preprint* astro-ph/0509422
[33] Sotiriou T P 2005 *Preprint* gr-qc/0509029
[34] Meng X H and Wang P 2003 *Preprint* astro-ph/0308284
[35] Olmo G J and Komp W 2004 *Preprint* gr-qc/0403092
[36] Koivisto T 2005 *Preprint* gr-qc/0505128
[37] Capozziello S, Cardone V F and Troisi A 2005 *Phys. Rev.* D **71** 043503