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The cosmic jerk parameter in $f(R)$ gravity

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Abstract

We derive the expression for the jerk parameter in $f(R)$ gravity. We use the Palatini variational principle and the field equations in the Einstein conformal gauge. For the particular case $f(R) = R - \frac{\alpha^2}{3R}$, the predicted value of the jerk parameter agrees with the SNLS SNIa and X-ray galaxy cluster distance data but does not with the SNIa gold sample data.

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1. Introduction

A particular class of alternative theories of gravity that has recently attracted a lot of interest is that of the $f(R)$ gravity models, in which the gravitational Lagrangian is a function of the curvature scalar R [1]. It has been shown that current cosmic acceleration may originate from the addition of a term R^{-1} to the Einstein–Hilbert Lagrangian R [2].

As in general relativity, $f(R)$ gravity theories obtain the field equations by varying the total action for both the field and matter. In this work we use the metric-affine (Palatini) variational principle, according to which the metric and connection are considered as geometrically independent quantities, and the action is varied with respect to both of them [3]. The other one is the metric (Einstein–Hilbert) variational principle, according to which the action is varied with respect to the metric tensor $g_{\mu\nu}$, and the affine connection coefficients are the Christoffel symbols of $g_{\mu\nu}$. Both approaches give the same result only if we use the standard Einstein–Hilbert action [4]. The field equations in the Palatini formalism are second-order differential equations, while for metric theories they are fourth-order. Another remarkable property of the metric-affine approach is that the field equations in vacuum reduce to the standard Einstein equations of general relativity with a cosmological constant [4].

One can show that $f(R)$ theories of gravitation are conformally equivalent to the Einstein theory of the gravitational field interacting with additional matter fields, if the action for matter does not depend on connection [3,5]. This can be done by means of a Legendre transformation, which in classical mechanics replaces the Lagrangian of a mechanical system with the Helmholtz Lagrangian. For $f(R)$ gravity, the scalar degree of freedom due to the occurrence of nonlinear second-order terms in the Lagrangian is transformed into an auxiliary scalar field ϕ [5]. The set of variables $(g_{\mu\nu}, \phi)$ is commonly called the *Jordan conformal gauge*. In the Jordan gauge, the connection is metric incompatible unless $f(R) = R$. The compatibility can be restored by a certain conformal transformation of the metric: $g_{\mu\nu} \to h_{\mu\nu} = f'(R)g_{\mu\nu}$. The new set $(h_{\mu\nu}, \phi)$ is called the *Einstein conformal gauge*, and we will regard the metric in this gauge as physical.

$f(R)$ gravity models have been compared with cosmological observations by several authors [6,7] and the problem of viability of these models is still open (see [8] and references therein). Current SNIa observations provide the data on the time evolution of the deceleration parameter q in the form of $q = q(z)$, where z is the redshift [9]. The extraction of the information from these data depends, however, on assumed parame-

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trization of $q(z)$ [10]. For small values of z such a dependence can be linear, $q(z) = q_0 + q_1 z$ [9], but its validity should fail at $z \sim 1$. A convenient method to describe models close to ΛCDM is based on the cosmic jerk parameter j, a dimensionless third derivative of the scale factor with respect to the cosmic time [11, 12]. A deceleration-to-acceleration transition occurs for models with a positive value of j_0 and negative q_0. Flat ΛCDM models have a constant jerk $j = 1$.

In this work we derive the general expression for the jerk parameter in $f(R)$ gravity. We use the field equations in the Palatini formalism and the Einstein conformal gauge [13]. We find the current value of this parameter for the case $f(R) = R - \frac{\alpha^2}{3R}$ [2,7] and compare it with recent cosmological data [10].

2. Palatini variation in $f(R)$ gravity

The action for $f(R)$ gravity in the original (Jordan) gauge with the metric $\tilde{g}_{\mu\nu}$ is given by [13]

$$S_J = -\frac{1}{2\kappa c}\int d^4x \left[\sqrt{-\tilde{g}} f(\tilde{R})\right] + S_m(\tilde{g}_{\mu\nu}, \psi). \tag{1}$$

Here, $\sqrt{-\tilde{g}} f(\tilde{R})$ is a Lagrangian density that depends on the curvature scalar $\tilde{R} = R_{\mu\nu}(\Gamma^\lambda_{\rho\sigma})\tilde{g}^{\mu\nu}$, S_m is the action for matter represented symbolically by ψ and independent of the connection, and $\kappa = \frac{8\pi G}{c^4}$. Tildes indicate quantities calculated in the Jordan gauge.

Variation of the action S_J with respect to $\tilde{g}_{\mu\nu}$ yields the field equations

$$f'(\tilde{R})R_{\mu\nu} - \frac{1}{2} f(\tilde{R})\tilde{g}_{\mu\nu} = \kappa T_{\mu\nu}, \tag{2}$$

where the dynamical energy–momentum tensor of matter is generated by the Jordan metric tensor:

$$\delta S_m = \frac{1}{2c}\int d^4x \sqrt{-\tilde{g}} T_{\mu\nu}\delta\tilde{g}^{\mu\nu}, \tag{3}$$

and the prime denotes the derivative of a function with respect to its variable. From variation of S_J with respect to the connection $\Gamma^\rho_{\mu\nu}$ it follows that this connection is given by the Christoffel symbols of the conformally transformed metric [5]

$$g_{\mu\nu} = f'(\tilde{R})\tilde{g}_{\mu\nu}. \tag{4}$$

The metric $g_{\mu\nu}$ defines the Einstein gauge, in which the connection is metric-compatible.

The action (1) is dynamically equivalent to the following Helmholtz action [5,13]:

$$S_H = -\frac{1}{2\kappa c}\int d^4x \sqrt{-\tilde{g}}\left[f\big(\phi(p)\big) + p\big(\tilde{R} - \phi(p)\big)\right] + S_m(\tilde{g}_{\mu\nu}, \psi), \tag{5}$$

where p is a new scalar variable. The function $\phi(p)$ is determined by

$$\left.\frac{\partial f(\tilde{R})}{\partial \tilde{R}}\right|_{\tilde{R}=\phi(p)} = p. \tag{6}$$

From Eqs. (4) and (6) it follows that

$$\phi = R f'(\phi), \tag{7}$$

where $R = R_{\mu\nu}(\Gamma^\lambda_{\rho\sigma})g^{\mu\nu}$ is the Riemannian curvature scalar of the metric $g_{\mu\nu}$.

In the Einstein gauge, the action (5) becomes the standard Einstein–Hilbert action of general relativity with an additional scalar field:

$$S_E = -\frac{1}{2\kappa c}\int d^4x \sqrt{-g}\left[R - p^{-1}\phi(p) + p^{-2} f\big(\phi(p)\big)\right] + S_m\big(p^{-1}g_{\mu\nu}, \psi\big). \tag{8}$$

Choosing ϕ (which is the curvature scalar in the Jordan gauge) as the scalar variable leads to

$$S_E = -\frac{1}{2\kappa c}\int d^4x \sqrt{-g}\left[R - 2V(\phi)\right] + S_m\big(\left[f'(\phi)\right]^{-1} g_{\mu\nu}, \psi\big), \tag{9}$$

where $V(\phi)$ is the effective potential

$$V(\phi) = \frac{\phi f'(\phi) - f(\phi)}{2[f'(\phi)]^2}. \tag{10}$$

Variation of the action (9) with respect to $g_{\mu\nu}$ yields the equations of the gravitational field in the Einstein gauge [13]:

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \frac{\kappa T_{\mu\nu}}{f'(\phi)} - V(\phi) g_{\mu\nu}, \tag{11}$$

while variation with respect to ϕ reproduces (7). Eqs. (7) and (11) give

$$\phi f'(\phi) - 2f(\phi) = \kappa T f'(\phi), \tag{12}$$

from which we obtain $\phi = \phi(T)$. Substituting ϕ into the field equations (11) leads to a relation between the Ricci tensor and the energy–momentum tensor. Such a relation is in general nonlinear and depends on the form of the function $f(R)$.

3. The jerk parameter in $f(R)$ gravity

The jerk parameter in cosmology is defined as [11,12]

$$j = \frac{\dddot{a}}{aH^3}, \tag{13}$$

where a is the cosmic scale factor, H is the Hubble parameter, and the dot denotes differentiation with respect to the cosmic time. This parameter appears in the fourth term of a Taylor expansion of the scale factor around a_0:

$$\frac{a(t)}{a_0} = 1 + H_0(t - t_0) - \frac{1}{2}q_0 H_0^2(t - t_0)^2 + \frac{1}{6} j_0 H_0^3(t - t_0)^3 + O\left[(t - t_0)^4\right], \tag{14}$$

where the subscript 0 denotes the present value. We can rewrite Eq. (13) as

$$j = q + 2q^2 - \frac{\dot{q}}{H}, \tag{15}$$

where q is the deceleration parameter. For flat ΛCDM model $j = 1$ [10].[1]

From the gravitational field equations (11) applied to a flat Robertson–Walker universe with dust we can derive the ϕ-dependence of the Hubble parameter [13]

$$H(\phi) = \frac{c}{f'(\phi)}\sqrt{\frac{\phi f'(\phi) - 3f(\phi)}{6}} \tag{16}$$

and the deceleration parameter [7]

$$q(\phi) = \frac{2\phi f'(\phi) - 3f(\phi)}{\phi f'(\phi) - 3f(\phi)}. \tag{17}$$

We also have the expression for the time dependence of ϕ [13]

$$\dot{\phi} = \frac{\sqrt{6}c(\phi f' - 2f)\sqrt{\phi f' - 3f}}{2f'^2 + \phi f' f'' - 6ff''}. \tag{18}$$

Combining Eqs. (16)–(18) and using $\dot{q} = \dot{\phi} q'(\phi)$ leads to

$$\frac{\dot{q}}{H} = \frac{18f'(\phi f' - 2f)(\phi f'^2 - \phi f f'' - ff')}{(\phi f' - 3f)^2(2f'^2 + \phi f' f'' - 6ff'')}. \tag{19}$$

From Eq. (15) we finally obtain

$$\begin{aligned} j(\phi) = {} & \big[2\phi^2 f'^4 + 10\phi^3 f'^3 f'' - 75\phi^2 f'^2 f f'' - 12\phi f f'^3 \\ & + 18 f^2 f'^2 + 189\phi f^2 f' f'' - 162 f^3 f''\big] \\ & \times \big[(\phi f' - 3f)^2(2f'^2 + \phi f' f'' - 6ff'')\big]^{-1}. \end{aligned} \tag{20}$$

We now examine the case $f(R) = R - \frac{\alpha^2}{3R}$, where α is a constant, which is a possible explanation of current cosmic acceleration [2]. In this model the present value of ϕ is $\phi_0 = (-1.05 \pm 0.01)\alpha$, where $\alpha = (7.35\pm^{1.12}_{1.17}) \times 10^{-52}\ \mathrm{m}^{-2}$ [7]. We do not need to know the exact value of α since it does not affect non-dimensional cosmological parameters. Substituting ϕ_0 into (20) gives

$$j_0 = 1.01^{+0.08}_{-0.21}. \tag{21}$$

This value does not overlap with the value $j = 2.16^{+0.81}_{-0.75}$, obtained from the combination of three kinematical data sets: the gold sample of type Ia supernovae [9], the SNIa data from the SNLS project [14], and the X-ray galaxy cluster distance measurements [10]. The origin of this disagreement could come from the assumption of constant jerk used there. However, two of the three data sets separately are consistent with the $f(R) = R - \frac{\alpha^2}{3R}$ model: the SNLS SNIa set gives $j = 1.32^{+1.37}_{-1.21}$ and the cluster set gives $j = 0.51^{+2.55}_{-2.00}$, and it is the gold sample data that yields larger $j = 2.75^{+1.22}_{-1.10}$ [10]. [2]

In the $f(R) = R - \frac{\alpha^2}{3R}$ model the deceleration-to-acceleration transition occurred at $\phi_t = -\sqrt{5/3}\alpha$ [7]. The cosmic jerk parameter at this moment can be found from Eq. (20):

$$j_t = \frac{10}{9}. \tag{22}$$

This value shows that the jerk parameter in $f(R)$ gravity changes significantly between the deceleration-to-acceleration transition and now, indicating the departure of $f(R)$ gravity models from ΛCDM. It would be interesting to generalize the kinematical approach of [10] to time dependent jerk and compare the results with $f(R)$ gravity models. More constraints on these models could also be provided by non-dimensional parameters containing higher derivatives of the scale factor, such as the snap parameter $s = \frac{\ddot{\ddot{a}}}{aH^4}$ [12].

[1] This identity can be easily verified from Eq. (15) for special cases where the deceleration parameter is constant: $q = 1/2$ (matter-dominated universe) and $q = -1$ (de Sitter universe).

[2] The value $q_0 = -0.81 \pm 0.14$ found in [10] from the combined three data sets agrees with $q_0 = -0.67\pm^{0.06}_{0.03}$ derived in the $f(R) = R - \frac{\alpha^2}{3R}$ model [7]. Each set separately agrees with our model as well.

4. Summary

We derived the expression for the cosmic jerk parameter in $f(R)$ gravity formulated in the Einstein gauge. We used the Palatini variational principle to obtain the field equations and apply them to a flat, homogeneous, and isotropic universe filled with dust. The value of the jerk parameter for the particular case $f(R) = R - \frac{\alpha^2}{3R}$ does not overlap with the value obtained from cosmological data of the SNIa gold sample, but is consistent with the values obtained from more recent SNLS SNIa data and the X-ray galaxy cluster data [10]. Therefore, Palatini $f(R)$ models in the Einstein gauge, including the case $f(R) = R - \frac{\alpha^2}{3R}$, provide a possible explanation of current cosmic acceleration. Further observations should give stronger constraints on j and on $f(R)$ gravity.

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