The cosmic jerk parameter in \( f(R) \) gravity

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Abstract

We derive the expression for the jerk parameter in \( f(R) \) gravity. We use the Palatini variational principle and the field equations in the Einstein conformal gauge. For the particular case \( f(R) = R - \frac{\alpha}{3} R^2 \), the predicted value of the jerk parameter agrees with the SNLS SNIa and X-ray galaxy cluster distance data but does not with the SNIa gold sample data.

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1. Introduction

A particular class of alternative theories of gravity that has recently attracted a lot of interest is that of the \( f(R) \) gravity models, in which the gravitational Lagrangian is a function of the curvature scalar \( R \) [1]. It has been shown that current cosmic acceleration may originate from the addition of a term \( R^{-1} \) to the Einstein–Hilbert Lagrangian [2].

As in general relativity, \( f(R) \) gravity theories obtain the field equations by varying the total action for both the field and matter. In this work we use the metric-affine (Palatini) variational principle, according to which the metric and connection are considered as geometrically independent quantities, and the action is varied with respect to both of them [3]. The other one is the metric (Einstein–Hilbert) variational principle, according to which the action is varied with respect to the metric tensor \( g_{\mu\nu} \), and the affine connection coefficients are the Christoffel symbols of \( g_{\mu\nu} \). Both approaches give the same result only if we use the standard Einstein–Hilbert action [4]. The field equations in the Palatini formalism are second-order differential equations, while for metric theories they are fourth-order. Another remarkable property of the metric-affine approach is that the field equations in vacuum reduce to the standard Einstein equations of general relativity with a cosmological constant [4].

One can show that \( f(R) \) theories of gravitation are conformally equivalent to the Einstein theory of the gravitational field interacting with additional matter fields, if the action for matter does not depend on connection [3,5]. This can be done by means of a Legendre transformation, which in classical mechanics replaces the Lagrangian of a mechanical system with the Helmholtz Lagrangian. For \( f(R) \) gravity, the scalar degree of freedom due to the occurrence of nonlinear second-order terms in the Lagrangian is transformed into an auxiliary scalar field \( \phi \) [5]. The set of variables \( (g_{\mu\nu}, \phi) \) is commonly called the Jordan conformal gauge. In the Jordan gauge, the connection is metric incompatible unless \( f(R) = R \). The compatibility can be restored by a certain conformal transformation of the metric: \( g_{\mu\nu} \rightarrow h_{\mu\nu} = f'(R)g_{\mu\nu} \). The new set \((h_{\mu\nu}, \phi)\) is called the Einstein conformal gauge, and we will regard the metric in this gauge as physical.

\( f(R) \) gravity models have been compared with cosmological observations by several authors [6,7] and the problem of viability of these models is still open (see [8] and references therein). Current SNIa observations provide the data on the time evolution of the deceleration parameter \( q \) in the form of \( q = q(z) \), where \( z \) is the redshift [9]. The extraction of the information from these data depends, however, on assumed parame-
trization of \( q(z) \) [10]. For small values of \( z \) such a dependence can be linear, \( q(z) = q_0 + q_1 z \) [9], but its validity should fail at \( z \sim 1 \). A convenient method to describe models close to \( \Lambda \) CDM is based on the cosmic jerk parameter \( j \), a dimensionless third derivative of the scale factor with respect to the cosmic time [11,12]. A deceleration-to-acceleration transition occurs for models with a positive value of \( j_0 \) and negative \( q_0 \). Flat \( \Lambda \) CDM models have a constant jerk \( j = 1 \).

In this work we derive the general expression for the jerk parameter in \( f(R) \) gravity. We use the field equations in the Palatini formalism and the Einstein conformal gauge [13]. We find the current value of this parameter for the case \( f(R) = R - \frac{\alpha^2}{3\kappa} \) [2,7] and compare it with recent cosmological data [10].

### 2. Palatini variation in \( f(R) \) gravity

The action for \( f(R) \) gravity in the original (Jordan) gauge with the metric \( \bar{g}_{\mu\nu} \) is given by [13]

\[
S_J = -\frac{1}{2\kappa c^4} \int d^4x \sqrt{-\bar{g}} f(\bar{R}) + S_m(\bar{g}_{\mu\nu}, \psi). \tag{1}
\]

Here, \( \sqrt{-\bar{g}} f(\bar{R}) \) is a Lagrangian density that depends on the curvature scalar \( \bar{R} = R_{\mu\nu}(\Gamma^\lambda_{\mu\nu})\bar{g}^{\lambda\nu} \), \( S_m \) is the action for matter represented symbolically by \( \psi \) and independent of the connection, and \( \kappa = \frac{8\pi G}{c^4} \). Tildes indicate quantities calculated in the Jordan gauge.

Variation of the action \( S_J \) with respect to \( \bar{g}_{\mu\nu} \) yields the field equations

\[
f'(\bar{R}) R_{\mu\nu} - \frac{1}{2} f(\bar{R}) \bar{g}_{\mu\nu} = \kappa T_{\mu\nu}, \tag{2}
\]

where the dynamical energy–momentum tensor of matter is generated by the Jordan metric tensor:

\[
\delta S_m = \frac{1}{2\kappa c^4} \int d^4x \sqrt{-\bar{g}} T_{\mu\nu} \delta \bar{g}^{\mu\nu}, \tag{3}
\]

and the prime denotes the derivative of a function with respect to its variable. From variation of \( S_J \) with respect to the connection \( \Gamma^\lambda_{\mu\nu} \) it follows that this connection is given by the Christoffel symbols of the conformally transformed metric [5]

\[
g_{\mu\nu} = f'(\bar{R}) \bar{g}_{\mu\nu}. \tag{4}
\]

The metric \( g_{\mu\nu} \) defines the Einstein gauge, in which the connection is metric-compatible.

The action (1) is dynamically equivalent to the following Helmholtz action [5,13]:

\[
S_H = -\frac{1}{2\kappa c^4} \int d^4x \sqrt{-g} \left[ f(\phi(p)) + p(\bar{R} - \phi(p)) \right] + S_m(\bar{g}_{\mu\nu}, \psi), \tag{5}
\]

where \( p \) is a new scalar variable. The function \( \phi(p) \) is determined by

\[
\frac{\partial f(\bar{R})}{\partial \bar{R}} \bigg|_{\bar{R}=\phi(p)} = p. \tag{6}
\]

From Eqs. (4) and (6) it follows that

\[
\phi = Rf'(\phi), \tag{7}
\]

where \( R = R_{\mu\nu}(\Gamma^\lambda_{\mu\nu})g^{\mu\nu} \) is the Riemannian curvature scalar of the metric \( g_{\mu\nu} \).

In the Einstein gauge, the action (5) becomes the standard Einstein–Hilbert action of general relativity with an additional scalar field:

\[
S_E = -\frac{1}{2\kappa c^4} \int d^4x \sqrt{-g} \left[ R - p^{-1} \phi(p) + p^{-2} f(\phi(p)) \right] + S_m(p^{-1} g_{\mu\nu}, \psi). \tag{8}
\]

Choosing \( \phi \) (which is the curvature scalar in the Jordan gauge) as the scalar variable leads to

\[
S_E = -\frac{1}{2\kappa c^4} \int d^4x \sqrt{-g} \left[ R - 2V(\phi) \right] + S_m \left[ f'(\phi) \right]^{-\frac{1}{2}} g_{\mu\nu}, \tag{9}
\]

where \( V(\phi) \) is the effective potential

\[
V(\phi) = \frac{\phi f'(\phi) - f(\phi)}{[f'(\phi)]^2}. \tag{10}
\]

Variation of the action (9) with respect to \( g_{\mu\nu} \) yields the equations of the gravitational field in the Einstein gauge [13]:

\[
R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \kappa T_{\mu\nu} - V(\phi) g_{\mu\nu}, \tag{11}
\]

while variation with respect to \( \phi \) reproduces (7). Eqs. (7) and (11) give

\[
\phi f'(\phi) - 2 f(\phi) = \kappa T f'(\phi), \tag{12}
\]

from which we obtain \( \phi = \phi(T) \). Substituting \( \phi \) into the field equations (11) leads to a relation between the Ricci tensor and the energy–momentum tensor. Such a relation is in general non-linear and depends on the form of the function \( f(R) \).

### 3. The jerk parameter in \( f(R) \) gravity

The jerk parameter in cosmology is defined as [11,12]

\[
j = \frac{\dot{a}}{aH^3}, \tag{13}
\]

where \( a \) is the cosmic scale factor, \( H \) is the Hubble parameter, and the dot denotes differentiation with respect to the cosmic time. This parameter appears in the fourth term of a Taylor expansion of the scale factor around \( a_0 \):

\[
a(t) = a_0(t - t_0) \left( 1 + \frac{1}{2} H_0^2(t - t_0)^2 \right) - \frac{7}{6} H_0^3(t - t_0)^3 + O[(t - t_0)^4], \tag{14}
\]

where the subscript 0 denotes the present value. We can rewrite Eq. (13) as

\[
j = q + 2a^2 - \frac{\dot{a}}{H}, \tag{15}
\]
where \( q \) is the deceleration parameter. For flat \( \Lambda \text{CDM} \) model \( j = 1 \) \cite{10}.\(^1\)

From the gravitational field equations \((11)\) applied to a flat Robertson–Walker universe with dust we can derive the \( \phi \)-dependence of the Hubble parameter \cite{13}

\[
H(\phi) = \frac{c}{f'(\phi)} \sqrt{|\frac{f''(\phi) - 3f(\phi)}{6}}
\]

(16)

and the deceleration parameter \cite{7}

\[
q(\phi) = \frac{2\phi f'f''(\phi) - 3f(\phi)}{\phi f'(\phi) - 3f(\phi)}.
\]

(17)

We also have the expression for the time dependence of \( \phi \) \cite{13}

\[
\dot{\phi} = \sqrt{6c(\phi f' - 2f)} \sqrt{\frac{f'' f' - 3f}{2f'^2 + \phi f' f'' - 6ff''}}.
\]

(18)

Combining Eqs. (16)–(18) and using \( \dot{q} = \dot{\phi} q' \) (\( \phi \) leads to

\[
\frac{\dot{q}}{H} = \frac{18 f'' (f' - 2f)(f' f'' - \phi f'' - j f'))}{(f' - 3f)^2(2f'^2 + \phi f' f'' - 6ff'')}. \tag{19}
\]

From Eq. (15) we finally obtain

\[
j(\phi) = \frac{2\phi f' f'^4 + 10 \phi^3 f' f'' - 75 \phi^2 f'^2 f' f'' - 12 \phi f' f'^3}{18 f'^2 f'' - 189 \phi f' f'' f''' - 162 f^3 f'''}
\]

\[
\times \left[(f'' - 3f)^2(2f'^2 + \phi f' f'' - 6ff'')\right]^{-1}. \tag{20}
\]

We now examine the case \( f(R) = R - \frac{a^2}{3\pi} \), where \( \alpha \) is a constant, which is a possible explanation of current cosmic acceleration \cite{2}. In this model the present value of \( \phi \) is \( \phi_0 = (-1.05 \pm 0.01) \alpha \), where \( \alpha = (7.35 \pm 1.17) \times 10^{-55} \text{ m}^{-2} \) \cite{7}. We do not need to know the exact value of \( \alpha \) since it does not affect non-dimensional cosmological parameters. Substituting \( \phi_0 \) into (20) gives

\[
j_0 = 1.01^{+0.08}_{-0.21} \tag{21}
\]

This value does not overlap with the value \( j = 2.16^{+0.81}_{-0.75} \) obtained from the combination of three kinematical data sets: the gold sample of type Ia supernovae \cite{9}, the SNIa data from the SNLS project \cite{14}, and the X-ray galaxy cluster distance measurements \cite{10}. The origin of this disagreement could come from the assumption of constant jerk used there. However, two of the three data sets separately are consistent with the \( f(R) = R - \frac{a^2}{3\pi} \) model: the SNLS SNIa set gives \( j = 1.32^{+1.37}_{-1.21} \) and the cluster set gives \( j = 0.51^{+2.55}_{-2.00} \) and it is the gold sample data that yields larger \( j = 2.75^{+1.22}_{-1.10} \) \cite{10}.\(^2\)

In the \( f(R) = R - \frac{a^2}{3\pi} \) model the deceleration-to-acceleration transition occurred at \( \phi_t = -\sqrt{5/3} \alpha \) \cite{7}. The cosmic jerk parameter at this moment can be found from Eq. (20):

\[
j_t = \frac{10}{9}. \tag{22}
\]

This value shows that the jerk parameter in \( f(R) \) gravity changes significantly between the deceleration-to-acceleration transition and now, indicating the departure of \( f(R) \) gravity models from \( \Lambda \text{CDM} \). It would be interesting to generalize the kinematical approach of \cite{10} to time dependent jerk and compare the results with \( f(R) \) gravity models. More constraints on these models could also be provided by non-dimensional parameters containing higher derivatives of the scale factor, such as the snap parameter \( s = \frac{\ddot{a}}{aH^2} \) \cite{12}.

4. Summary

We derived the expression for the cosmic jerk parameter in \( f(R) \) gravity formulated in the Einstein gauge. We used the Palatini variational principle to obtain the field equations and apply them to a flat, homogeneous, and isotropic universe filled with dust. The value of the jerk parameter for the particular case \( f(R) = R - \frac{a^2}{3\pi} \) does not overlap with the value obtained from cosmological data of the SNIa gold sample, but is consistent with the values obtained from more recent SNLS SNIa data and the X-ray galaxy cluster data \cite{10}. Therefore, Palatini \( f(R) \) models in the Einstein gauge, including the case \( f(R) = R - \frac{a^2}{3\pi} \), provide a possible explanation of current cosmic acceleration. Further observations should give stronger constraints on \( j \) and \( f(R) \) gravity.

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\(^1\) This identity can be easily verified from Eq. (15) for special cases where the deceleration parameter is constant: \( q = 1/2 \) (matter-dominated universe) and \( q = -1 \) (de Sitter universe).

\(^2\) The value \( q_0 = -0.81 \pm 0.14 \) found in \cite{10} from the combined three data sets agrees with \( q_0 = -0.67^{+0.09}_{-0.03} \) derived in the \( f(R) = R - \frac{a^2}{3\pi} \) model \cite{7}. Each set separately agrees with our model as well.