

**Interacting dark energy in  $f(R)$  gravity**

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The field equations in  $f(R)$  gravity derived from the Palatini variational principle and formulated in the Einstein conformal frame yield a cosmological term which varies with time. Moreover, they break the conservation of the energy-momentum tensor for matter, generating the interaction between matter and dark energy. Unlike phenomenological models of interacting dark energy,  $f(R)$  gravity derives such an interaction from a covariant Lagrangian which is a function of a relativistically invariant quantity (the curvature scalar  $R$ ). We derive the expressions for the quantities describing this interaction in terms of an arbitrary function  $f(R)$ , and examine how the simplest phenomenological models of a variable cosmological constant are related to  $f(R)$  gravity. Particularly, we show that  $\Lambda c^2 = H^2(1 - 2q)$  for a flat, homogeneous and isotropic, pressureless universe. For the Lagrangian of form  $R - 1/R$ , which is the simplest way of introducing current cosmic acceleration in  $f(R)$  gravity, the predicted matter-dark energy interaction rate changes significantly in time, and its current value is relatively weak (on the order of 1% of  $H_0$ ), in agreement with astronomical observations.

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**I. INTRODUCTION**

Einstein's general relativity is based on the Lagrangian that is a linear function of the Riemann curvature scalar. The linearity of the gravitational action with respect to the curvature results in the Einstein equations of the gravitational field that are second-order differential equations for the metric tensor [1]. The Einstein equations applied to a homogeneous and isotropic spacetime lead to the Friedmann equations describing an expanding universe with a positive deceleration. The most accepted explanation of the observed cosmic acceleration [2] is that the universe is dominated by dark energy (quintessence) [3,4]. The simplest way of introducing dark energy into general relativity is to add a cosmological constant  $\Lambda$  to the curvature scalar  $R$  in the Lagrangian (the  $\Lambda$ CDM model):

$$S_g = -\frac{1}{2\kappa c} \int d^4x \sqrt{-g} [R + 2\Lambda]. \quad (1)$$

However, it is also possible to modify the curvature dependence of the gravitational action to obtain the field equations that allow accelerated expansion. A particular class of alternative theories of gravity that has recently attracted a lot of interest is that of the  $f(R)$  gravity models, in which the gravitational Lagrangian is a nonlinear function of  $R$  [5]. It has been shown that current cosmic acceleration may originate from the addition of a term  $R^{-1}$  (or other negative powers of  $R$ ) to the Einstein-Hilbert Lagrangian  $R$  [6]. As in general relativity,  $f(R)$  gravity theories obtain the field equations by varying the total action for both the field and matter.

In the literature, there are two approaches on how to perform the variation. We use the metric-affine (Palatini) variational principle, according to which the metric and connection are considered as geometrically independent quantities, and the action is varied with respect to both of

them. The other one is the metric (Einstein-Hilbert) variational principle, according to which the action is varied with respect to the metric tensor  $g_{\mu\nu}$ , and the affine connection coefficients are the Christoffel symbols of  $g_{\mu\nu}$ . The field equations in the Palatini formalism are second-order differential equations, while for metric theories they are fourth-order. Both approaches give the same result only if we use the standard Einstein-Hilbert action [7].

An interaction between ordinary matter and dark energy has been introduced in the form of a time dependent cosmological constant  $\Lambda(t)$  [8]. A decreasing (decaying)  $\Lambda$  gives a positive rate of the entropy production and could explain the observed large entropy of the universe. The cosmological constant can depend on the cosmic time directly [9] or through other cosmological variables such as the scale factor  $a$  or the Hubble parameter  $H$ . Dimensional arguments lead to  $\Lambda \propto a^{-2}$  [10,11],  $\Lambda \propto H^2$  [12], or both [13]. A more general form  $\Lambda \propto a^{-m}$ , where  $m = \text{const}$  [14], is predicted by holographic cosmology [15–17]. The cosmological term can also involve other quantities such as the deceleration parameter or temperature. Ref. [18] lists and reviews phenomenological models with a variable cosmological term, and examines evolution of the scale factor in these models.

The field equations in  $f(R)$  gravity formulated in the Einstein conformal frame not only yield an effective cosmological term, but also break the conservation of the energy-momentum tensor for matter [19], generating the interaction between matter and dark energy. Therefore,  $f(R)$  gravity provides a physical explanation for the exchange of energy between cosmological term and other forms of matter. Unlike phenomenological models of interacting dark energy, and like Brans-Dicke cosmologies with the cosmological term depending on the scalar field [20], it derives such an interaction from a covariant Lagrangian in which the cosmological constant is a func-

tion of a relativistically invariant quantity ( $R$ ). Furthermore,  $f(R)$  gravity in the Einstein frame generates a variation of the gravitational constant with the cosmological time, as in the large number hypothesis of Dirac [21].

The aim of this paper is to find how this interaction depends on the function  $f(R)$ . In Sec. II we derive the equations of field in  $f(R)$  gravity using the Palatini variational principle. In Sec. III we apply these equations to a homogeneous and isotropic, flat universe filled with dust, and obtain the expressions describing the interaction between matter and dark energy. We also examine how the simplest phenomenological models of a variable cosmological constant are related to  $f(R)$  gravity. In Sec. IV we give numerical predictions for the case  $f(R) = R + \text{const} \times R^{-1}$ , which is the simplest way of introducing current cosmic acceleration in  $f(R)$  gravity [6], and compare them with other models of interacting dark energy. The results are summarized in Sec. V.

## II. THE FIELD EQUATIONS

The action for an  $f(R)$  gravity is given by [19,22]

$$S_J = -\frac{1}{2\kappa c} \int d^4x [\sqrt{-\tilde{g}} f(\tilde{R})] + S_m(\tilde{g}_{\mu\nu}, \psi). \quad (2)$$

Here,  $\sqrt{-\tilde{g}} f(\tilde{R})$  is a Lagrangian density that depends on the curvature scalar  $\tilde{R} = R_{\mu\nu}(\Gamma_{\rho\sigma}^\lambda) \tilde{g}^{\mu\nu}$ ,  $S_m$  is the action for matter represented symbolically by  $\psi$  and assumed to be independent of the connection  $\Gamma_{\rho\sigma}^\lambda$ , and the connection is assumed to be symmetric (no torsion). The variables describing an  $f(R)$  Lagrangian are said to form the Jordan conformal frame (*JCF*) [23]. Tildes indicate quantities calculated in this frame, e.g.,  $\tilde{g}_{\mu\nu}$  is the JCF metric tensor.

Variation of the action  $S_J$  with respect to  $\tilde{g}_{\mu\nu}$  yields the field equations

$$f'(\tilde{R}) R_{\mu\nu} - \frac{1}{2} f(\tilde{R}) \tilde{g}_{\mu\nu} = \kappa T_{\mu\nu}, \quad (3)$$

where the dynamical energy-momentum tensor of matter  $T_{\mu\nu}$  is generated by the JCF metric tensor:

$$\delta S_m = \frac{1}{2c} \int d^4x \sqrt{-\tilde{g}} T_{\mu\nu} \delta \tilde{g}^{\mu\nu}. \quad (4)$$

The prime denotes the derivative of the function  $f(\tilde{R})$  with respect to  $\tilde{R}$ . From variation of  $S_J$  with respect to the connection  $\Gamma_{\mu\nu}^\rho$  it follows that this connection is given by the Christoffel symbols of the conformally transformed metric [5]

$$g_{\mu\nu} = f'(\tilde{R}) \tilde{g}_{\mu\nu}. \quad (5)$$

The metric  $g_{\mu\nu}$  defines the Einstein conformal frame (*ECF*), in which the connection is metric compatible [23].

A transition from the JCF (modified gravity) to the ECF (general relativity) in  $f(R)$  gravity is possible if  $S_m$  does not contain torsion [22]. Such a transition is also possible

for more general theories in which the gravitational Lagrangian depends on the Ricci tensor, but not on the Weyl tensor [24]. In the Jordan frame, the connection is metric incompatible unless  $f(R) = R$ . We regard the ECF metric tensor as physical, although whether it is true or not should be ultimately decided by experiment or observation. Since the physical equivalence between both frames is an open problem as well [25], so is the answer to the question whether and how the results of this paper would change if the JCF was physical.

In the Einstein frame, the action (2) becomes the standard general-relativistic action of the gravitational field interacting with an additional nondynamical scalar field [19,26]:

$$S_E = -\frac{1}{2\kappa c} \int d^4x \sqrt{-g} [R - 2V(\tilde{R})] + S_m([f'(\tilde{R})]^{-1} g_{\mu\nu}, \psi), \quad (6)$$

where  $R = R_{\mu\nu}(\Gamma_{\rho\sigma}^\lambda) g^{\mu\nu}$  is the Riemannian curvature scalar of the metric  $g_{\mu\nu}$ , and  $V(\tilde{R})$  is the effective potential,

$$V(\tilde{R}) = \frac{\tilde{R} f'(\tilde{R}) - f(\tilde{R})}{2[f'(\tilde{R})]^2}. \quad (7)$$

The curvature scalars in both frames are related by

$$R = f'(\tilde{R}) \tilde{R}, \quad (8)$$

which follows from (5).

Variation of the action (6) with respect to  $g_{\mu\nu}$  gives the equation of the gravitational field in the Einstein frame [19,27]:

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \frac{\kappa T_{\mu\nu}}{f'(\tilde{R})} - V(\tilde{R}) g_{\mu\nu}. \quad (9)$$

Eqs. (8) and (9) yield an algebraic relation

$$\tilde{R} f'(\tilde{R}) - 2f(\tilde{R}) = \kappa T f'(\tilde{R}), \quad (10)$$

from which we obtain  $\tilde{R}$  as a function of  $T = T_{\mu\nu} g^{\mu\nu}$  for a given  $f(\tilde{R})$  [28]. If  $T = 0$  (vacuum or radiation) then  $\tilde{R} = \text{const}$  and the solution of the field equation is an empty spacetime with a cosmological constant [7]. Substituting  $\tilde{R}$  into (9) leads to a final relation between the geometrical tensors and the energy-momentum tensor,

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \kappa_r(T) T_{\mu\nu} + \Lambda(T) g_{\mu\nu}, \quad (11)$$

with a running gravitational coupling  $\kappa_r(T) = \kappa [f'(\tilde{R}(T))]^{-1}$  and a variable cosmological term  $\Lambda(T) = -V(\tilde{R}(T))$ :

$$\Lambda(\tilde{R}) = \frac{f(\tilde{R}) - \tilde{R} f'(\tilde{R})}{2[f'(\tilde{R})]^2}. \quad (12)$$

Such a relation is, in general, nonlinear and depends on the form of the function  $f(\tilde{R})$ . Moreover, the time dependence of  $T$  makes the quantities  $\kappa_r$  and  $\Lambda$  variable, although they

are not associated with any dynamical quintessence scalar field as in [3,29]. Concluding, Palatini  $f(R)$  gravity does not change the structure of Einstein's general relativity but introduces a varying gravitational constant [21], as well as a term acting like a cosmological constant which varies with time. It modifies the gravitational field inside material objects, while the metric in vacuum (such as the Schwarzschild solution for spherically symmetric systems) is the same as that in general relativity.

### III. THE INTERACTION BETWEEN MATTER AND DARK ENERGY

The Bianchi identity applied to (9) gives

$$T_{\mu\nu}{}^{;\nu} = \tilde{R}{}^{\nu}{}_{\nu} f''(\tilde{R}) \left( \frac{T_{\mu\nu}}{f'(\tilde{R})} + \frac{[2f(\tilde{R}) - \tilde{R}f'(\tilde{R})]g_{\mu\nu}}{2\kappa[f'(\tilde{R})]^2} \right). \quad (13)$$

This relation means that the energy-momentum tensor in the Einstein frame is *not* covariantly conserved, unless  $f(R) = R$  or  $T = 0$  [30]. We can write the field Eq. (9) as

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \kappa(T_{\mu\nu}^m + T_{\mu\nu}^\Lambda), \quad (14)$$

where  $T_{\mu\nu}^m = T_{\mu\nu}$ . This defines the dark energy-momentum tensor,

$$T_{\mu\nu}^\Lambda = \frac{\Lambda(\tilde{R})}{\kappa} g_{\mu\nu} + \frac{1 - f'(\tilde{R})}{f'(\tilde{R})} T_{\mu\nu}. \quad (15)$$

From Eq. (14) it follows that matter and dark energy form together a system that has a conserved four-momentum. Therefore, we can speak of an *interaction* between matter and dark energy [31]. In the Jordan frame of  $f(R)$  gravity, the energy-momentum tensor is always conserved [32] as a direct consequence of Eq. (4) for the case of an infinitesimal translation [1].

For a flat [33] Robertson-Walker universe with pressureless matter (dust)  $p_m = 0$  and in the comoving frame of reference, Eq. (13) gives the first Friedmann equation,

$$f'(\tilde{R}) \frac{d}{dt} \left[ \frac{\epsilon_m(\tilde{R})}{f'(\tilde{R})} - \frac{V(\tilde{R})}{\kappa} \right] + 3H(\tilde{R})\epsilon_m(\tilde{R}) = 0, \quad (16)$$

where the Hubble parameter is given by [19]

$$H(\tilde{R}) = \frac{c}{f'(\tilde{R})} \sqrt{\frac{\tilde{R}f'(\tilde{R}) - 3f(\tilde{R})}{6}}. \quad (17)$$

The matter energy density  $\epsilon_m(\tilde{R}) = T(\tilde{R})$  is obtained from (10),

$$\epsilon_m = \frac{\tilde{R}f'(\tilde{R}) - 2f(\tilde{R})}{\kappa f'(\tilde{R})}. \quad (18)$$

Eqs. (16) and (17) give the time evolution of  $\tilde{R}$ ,

$$\dot{\tilde{R}} = \frac{\sqrt{6c}(\tilde{R}f' - 2f)\sqrt{\tilde{R}f' - 3f}}{2f'^2 + \tilde{R}f'f'' - 6ff''}, \quad (19)$$

and we wrote  $f = f(\tilde{R})$  for brevity. The equation of continuity (16) can be written as

$$\dot{\epsilon}_m + 3H\epsilon_m = Q, \quad (20)$$

where

$$Q = \frac{\dot{\tilde{R}}f''(\tilde{R})[\tilde{R}f'(\tilde{R}) - 2f(\tilde{R})]}{2\kappa[f'(\tilde{R})]^2}. \quad (21)$$

The quantity  $Q$  describes the interaction between matter and dark energy [34], and vanishes for the general-relativistic case  $f(R) = R$ .

In the comoving frame of reference, Eq. (15) reads

$$\epsilon_\Lambda = \frac{\Lambda}{\kappa} + \frac{1 - f'(\tilde{R})}{f'(\tilde{R})} \epsilon_m, \quad (22)$$

$$p_\Lambda = -\frac{\Lambda}{\kappa}. \quad (23)$$

The definitions in Eq. (15) were chosen so that  $\Omega_m + \Omega_\Lambda = \epsilon_m/\epsilon_c + \epsilon_\Lambda/\epsilon_c = 1$ , where  $\epsilon_c = 3H^2/(\kappa c^2)$  is the critical energy density,

$$\epsilon_c = \frac{\tilde{R}f'(\tilde{R}) - 3f(\tilde{R})}{2\kappa[f'(\tilde{R})]^2}. \quad (24)$$

We also find the equation of continuity for dark energy,

$$\dot{\epsilon}_\Lambda + 3H(\epsilon_\Lambda + p_\Lambda) = -Q, \quad (25)$$

which has the same form as in [34]. Eqs. (20) and (25) are a realization of the fact that the system matter-dark energy is closed. The interaction term  $Q$  corresponds to the production of particles from quintessence.

In  $f(R)$  gravity, the dark energy density is given by

$$\epsilon_\Lambda = \frac{\tilde{R}f'(\tilde{R}) - 3f(\tilde{R}) - 2f'(\tilde{R})[\tilde{R}f'(\tilde{R}) - 2f(\tilde{R})]}{2\kappa[f'(\tilde{R})]^2}. \quad (26)$$

Consequently, the rate of interaction  $\Gamma = Q/\epsilon_\Lambda$  [34] equals

$$\Gamma = \frac{\sqrt{6c}f''(\tilde{R}f' - 2f)\sqrt{\tilde{R}f' - 3f}}{[\tilde{R}f' - 3f - 2f'(\tilde{R}f' - 2f)](2f'^2 + \tilde{R}f'f'' - 6ff'')}, \quad (27)$$

and the nondimensional ratio  $\gamma = \Gamma/H$  takes the form

$$\gamma = \frac{6f'f''(\tilde{R}f' - 2f)^2}{[\tilde{R}f' - 3f - 2f'(\tilde{R}f' - 2f)](2f'^2 + \tilde{R}f'f'' - 6ff'')}. \quad (28)$$

Finally, the ratio  $w_\Lambda = p_\Lambda/\epsilon_\Lambda$  is given by

$$w_\Lambda = \frac{\tilde{R}f' - f}{\tilde{R}f' - 3f - 2f'(\tilde{R}f' - 2f)}. \quad (29)$$

The case  $w_\Lambda = \text{const}$  was examined, for example, in [35]. In  $f(R)$  gravity, both  $\gamma$  and  $w_\Lambda$  are *not* constant, in general.

We now examine the consistence of the  $f(R)$  description of interacting dark energy with some of the phenomenological models studied in [18]. We begin with the decay law  $\Lambda = \beta H^2/c^2$ , where  $\beta$  is a nondimensional and positive (since  $\Lambda$  decreases with time) constant. This condition follows from the dimensional relations  $t_H \sim H^{-1}$  and  $\Lambda \sim l_{\text{Pl}}^{-2}(t_{\text{Pl}}/t_H)^2$ , where  $t_H$  is the Hubble time, and  $l_{\text{Pl}}$  and  $t_{\text{Pl}}$  are, respectively, the Planck length and time. It can also be deduced from Dirac's large number hypothesis [12], and the holographic principle [36]. Using Eqs. (12) and (17) we obtain

$$\frac{\tilde{R}f'(\tilde{R}) - f(\tilde{R})}{\tilde{R}f'(\tilde{R}) - 3f(\tilde{R})} = -\frac{\beta}{3}, \quad (30)$$

which has the solution in the form of a power function,  $f(\tilde{R}) \sim \tilde{R}^{3(1+\beta)/(3+\beta)}$ . Therefore, the relation  $\Lambda \propto H^2$  is inconsistent with  $f(R)$  gravity models which assume  $f(R) = R + \varepsilon g(R)$ , where  $g(R)$  is some nonlinear function of  $R$  and  $\varepsilon$  is a small quantity. Such a form of  $f(R)$  is required for the deviations from general relativity to be compatible with solar system experiments [37].

Considering  $\Lambda \propto H^2$  is equivalent [12], at the level of the Einstein equations in general-relativistic cosmology, to the constancy of the parameter [38]

$$x = \frac{\epsilon_\Lambda}{\epsilon_m + \epsilon_\Lambda}, \quad (31)$$

i.e. the constancy of the ratio  $r = \epsilon_m/\epsilon_\Lambda$  [39,40]. This is not the case in  $f(R)$  gravity, where this parameter is given by

$$x = \frac{\tilde{R}f' - 3f - 2f'(\tilde{R}f' - 2f)}{\tilde{R}f' - 3f}, \quad (32)$$

and its constancy leads to a first-order differential equation for  $f(\tilde{R})$  which has no power-function solutions if  $x \neq 0$  and  $x \neq 1$  [41]. A more general case  $r \propto a^{-\xi}$ , where  $\xi = \text{const}$ , was examined in [42]. The condition  $\Lambda \propto H^2$  has also been used in [43], together with the constancy of  $\gamma$  (and  $r$ ). The latter leads to a complicated, nonlinear second-order differential equation for the function  $f(\tilde{R})$ . A similar equation is obtained for the decay law  $\gamma \propto r$  used in [35,44]. We conclude that  $f(R)$  gravity does not favor theories which constrain  $Q$ ,  $\Gamma$ , or  $\gamma$ . Lastly, we note that a more general condition  $\Lambda \propto H^n$ , where  $n = \text{const} \neq 2$ , has no power-function solutions for  $f(\tilde{R})$  as well.

From the dependence of the redshift on  $\tilde{R}$  [30] we obtain

$$a^{-3} = \frac{b[\tilde{R}f'(\tilde{R}) - 2f(\tilde{R})]}{[f'(\tilde{R})]^2}, \quad (33)$$

where the constant  $b$  is given by

$$b = \frac{[f'(\tilde{R}_0)]^2}{a_0^3[\tilde{R}_0 f'(\tilde{R}_0) - 2f(\tilde{R}_0)]}, \quad (34)$$

and the subscript 0 denotes the corresponding present value. Therefore, the decay law  $\Lambda \propto a^{-m}$ , where  $m = \text{const}$ , yields

$$\frac{[f'(\tilde{R})]^{(2m/3)-2}[f(\tilde{R}) - \tilde{R}f'(\tilde{R})]}{[\tilde{R}f'(\tilde{R}) - 2f(\tilde{R})]^{m/3}} = \delta = \text{const}. \quad (35)$$

If  $m = 3$  then Eq. (35) has the power-function solution,  $f(\tilde{R}) \sim \tilde{R}^{(1-2\delta)/(1-\delta)}$ . The positivity of  $\epsilon_\Lambda$  and  $\Lambda$  requires  $\delta < 0$ . For  $m = 2$  [10,11],  $m = 4$  [14,17,38], or other values of  $m$  [18], Eq. (35) is quite complicated.

We now express the condition  $\Lambda \propto q$  [18], where  $q$  is the deceleration parameter, in  $f(R)$  gravity. Using the formula for  $q$  [30],

$$q(\tilde{R}) = \frac{2\tilde{R}f'(\tilde{R}) - 3f(\tilde{R})}{\tilde{R}f'(\tilde{R}) - 3f(\tilde{R})}, \quad (36)$$

we find

$$\frac{(\tilde{R}f' - f)(\tilde{R}f' - 3f)}{f'^2(2\tilde{R}f' - 3f)} = \text{const}. \quad (37)$$

The only power function obeying this condition is  $f(\tilde{R}) = \tilde{R}^2$ , which is excluded by Eq. (10). Concluding, there is no simple correspondence between  $f(R)$  gravity and the phenomenological, power-law models examined in [18] which relate the cosmological constant to any (one) cosmological quantity from the set  $(H, q, a)$ . In order to find the function  $f(\tilde{R})$  which corresponds to such an expression for  $\Lambda$ , we need to solve a complicated, nonlinear differential equation.

However, using Eqs. (17), (33), and (36), we can compose the exact cosmological term (12) from any two cosmological parameters from the set  $(H, q, a)$ , which gives three combinations. We find the following relations:

$$\Lambda = \frac{H^2}{c^2}(1 - 2q), \quad (38)$$

$$\Lambda = \frac{3H^2}{c^2} - \frac{a^{-3}}{b}, \quad (39)$$

$$\Lambda = \frac{1 - 2q}{2(1 + q)} \frac{a^{-3}}{b}. \quad (40)$$

All three are satisfied in  $f(R)$  gravity, but we only need one to compare  $f(R)$  gravity with a particular phenomenological model. The first equation generalizes the law  $\Lambda \propto H^2$  in the sense that the proportionality constant depends on the deceleration parameter,  $\beta = 1 - 2q$ . Equation (38) has the same form for an arbitrary function  $f(R)$ , and is an important prediction of the Palatini  $f(R)$  gravity [45]. The two other relations involve the constant  $b$  which depends on the function  $f(\tilde{R})$  and the present value  $\tilde{R}_0$ , i.e. they are

model dependent. Equation (39) is similar to the form assumed in [13], except we have  $a^{-3}$  instead of  $a^{-2}$ . We note that if  $\Lambda$  depends on  $q$  then the relation  $q = 1/2$ , which holds for the matter era, yields  $\Lambda = 0$ .

In general-relativistic cosmology, some phenomenological laws are dynamically equivalent to each other [46,47]. For example, the relations  $\Lambda \propto H^2$ ,  $\Lambda \propto qH^2$  [48], and  $\Lambda \propto \epsilon_m$  [49] give the same cosmological evolution [47]. Since the decay law  $\Lambda \propto H^2$  appears to be inconsistent with current constraints on the dark energy equation of state [16,46], so do the other equivalent models. In  $f(R)$  gravity, the condition  $\Lambda = \mu q H^2 / c^2$ , where  $\mu = \text{const}$ , becomes

$$\frac{\tilde{R}f'(\tilde{R}) - f(\tilde{R})}{2\tilde{R}f'(\tilde{R}) - 3f(\tilde{R})} = -\frac{\mu}{3}. \quad (41)$$

Equation (41) has the solution in the form of a power function,  $f(\tilde{R}) \sim \tilde{R}^{3(1+\mu)/(3+2\mu)}$ . This solution is equivalent to that of Eq. (30) if we associate  $\mu = 2\beta/(1-\beta)$ , in agreement with [46,47]. On the other hand, the condition  $\Lambda \propto \epsilon_m$  leads in  $f(R)$  gravity to

$$\frac{\tilde{R}f' - f}{f'(\tilde{R}f' - 2f)} = \text{const}, \quad (42)$$

which has no power-function solutions and is not equivalent to the law  $\Lambda \propto H^2$ .

Finally, we should mention that the decay law (38) predicted by  $f(R)$  gravity resembles the form used in [50],  $\Lambda \propto R = -6H^2(1-q)/c^2$ . This form is also equivalent, in general-relativistic cosmology, to the condition  $\Lambda \propto H^2$  [46]. Reference [46] showed that almost all the current phenomenological models of decaying vacuum (dark energy) can be unified in the form of the modified matter expansion rate,  $\epsilon_m \propto a^{-3+\varepsilon}$ , where  $\varepsilon = \text{const}$  [51]. The relation  $\Lambda \propto H^2$  requires  $\varepsilon > 1$  to get a currently accelerating universe. For example, the results of [43] give  $\varepsilon_0 = 1.85$  [52]. However, this inequality leads also to an accelerating expansion of the matter dominated universe. Overall, the type Ia supernovae, cosmic microwave background, and large-scale structure observations give the constraint  $\varepsilon < 0.1$ , which rules out the law  $\Lambda \propto H^2$  [16,46].

#### IV. THE $R - 1/R$ GRAVITY

The consistence of  $f(R)$  models with cosmological data has been studied for both metric and Palatini variational formalisms [53]. We examine a particular case

$$f(R) = R - \frac{\alpha^2}{3R}, \quad (43)$$

where  $\alpha$  is a constant, which is the simplest way to introduce current cosmic acceleration in  $f(R)$  gravity [6]. This model is referred to as the  $R - 1/R$  gravity, and appears (with Palatini variation and in the ECF) to be

compatible with cosmological data, although more observations are necessary to put stronger constraints on the analyzed parameters [54]. The current value of  $\tilde{R}$  equals  $\tilde{R}_0 = (-1.05 \pm 0.01)\alpha$  [30], which gives the present value of  $\gamma$ ,

$$\gamma_0 = 0.015 \pm 0.005. \quad (44)$$

Such a small value indicates that the interaction between matter and dark energy is *weak*, as compared to the direct interaction between dark energy and spacetime. A large error arises from the sensitivity of the term  $Rf'(\tilde{R}) - 2f(\tilde{R})$  around  $\tilde{R}_0$ .

Similarly, we find the value of  $\gamma$  at the deceleration-to-acceleration transition which occurred at  $\tilde{R}_t = -\sqrt{5/3}\alpha$  [30]:

$$\gamma_t = \frac{4}{15}. \quad (45)$$

The interaction rate drops significantly ( $\sim 20$  times) from the time of the transition to the present, which does not support the assumption of the constancy of  $\gamma$  [39,43]. As the universe approaches asymptotically a de Sitter expansion,  $\tilde{R}$  tends to  $-\alpha$  and the interaction rate  $\gamma$  between matter and dark energy decreases to zero.

In the case of the  $R - 1/R$  gravity, Eq. (29) for the present time gives

$$w_{\Lambda,0} = -1.10 \pm 0.02. \quad (46)$$

This result is consistent with the observed  $w_{\Lambda,0} = -1.02^{+0.13}_{-0.19}$  [33]. We also find the value of  $w_{\Lambda}$  at the deceleration-to-acceleration transition,

$$w_{\Lambda,t} = -\frac{5}{3}. \quad (47)$$

As the universe approaches a de Sitter phase,  $w_{\Lambda}$  tends to the value  $-1$ . Such a value corresponds to dark energy which does not interact with other forms of matter. Similarly, Eq. (32) reads

$$x_0 = 0.69^{+0.05}_{-0.04}, \quad (48)$$

$$x_t = \frac{1}{5}. \quad (49)$$

The observational constraint on the ratio  $x$  in the matter era is  $x < 3 \times 10^{-5}$  [38]. The deceleration-to-acceleration transition occurred when the universe was, clearly, dominated by dark energy. As the universe approaches a de Sitter expansion,  $x$  tends to 1.

From Eq. (38), we obtain the present value for the cosmological term  $\Lambda$ . Substituting the observed values  $H_0 = 71 \pm 4 \text{ km s}^{-1} \text{ Mpc}^{-1}$  [55] and  $q_0 = -0.74 \pm 0.18$  [33], we find

$$\Lambda_0 = (1.46^{+0.41}_{-0.35}) \times 10^{-52} \text{ m}^{-2}. \quad (50)$$

If we use the  $q_0$  predicted by the  $R - 1/R$  gravity,  $q_0 = -0.67^{+0.06}_{-0.03}$  [30], we arrive at

$$\Lambda_0 = (1.38_{-0.22}^{+0.20}) \times 10^{-52} \text{ m}^{-2}. \quad (51)$$

The cosmological constant at the deceleration-to-acceleration transition is given by  $\Lambda_t = H_t^2/c^2$ . Its value depends on the form of the function  $f(R)$ . For the case (43), we use the expressions for  $H_t$ ,  $\alpha$ , and  $q_0$  found in [30] to obtain

$$\frac{\Lambda_0}{\Lambda_t} = 0.96 \pm 0.08. \quad (52)$$

We see that  $\Lambda$  has not changed much since the time when  $q$  changed the sign. A significant decrease of the cosmological term must have happened earlier.

Using  $\Omega_{m,0} = 0.29_{-0.03}^{+0.05}$  [33] then we obtain

$$\frac{r_0}{\gamma_0} = 27.2_{-1.4}^{+7.9}. \quad (53)$$

This value differs significantly from that found in [43],  $r_0/\gamma_0 = 0.54$ . For the deceleration-to-acceleration transition (which is in the range of redshifts examined by [33]), we have  $\Omega_{m,t} = 4/5$  [30]. This gives

$$\frac{r_t}{\gamma_t} = 15, \quad (54)$$

which is again much larger than  $\sim 1$ . Therefore, the condition relating the dark energy density to the Hubble parameter together with the assumption of the constancy of the interaction rate are incompatible with the  $R - 1/R$  gravity. This conclusion agrees with the results of [46] which disfavor the decay law  $\Lambda \propto H^2$ , and the results of [42] which support  $\dot{r} \neq 0$ .

Equation (20) can be written as

$$\dot{\epsilon}_m + nH\epsilon_m = 0, \quad (55)$$

where

$$n = 3 - \gamma(\Omega_m^{-1} - 1). \quad (56)$$

The  $R - 1/R$  gravity predicts for the present time

$$n_0 = 2.96 \pm 0.01. \quad (57)$$

For the moment of the deceleration-to-acceleration transition, the  $R - 1/R$  gravity gives

$$n_t = 2.93. \quad (58)$$

The departure of  $n$  from 3 means that matter alone is not conserved. The above values of  $n$  are smaller than 3 (unlike 36/11 found in [56]), indicating that matter is being produced from dark energy. It has been shown [30] that the largest deviation from the standard nonrelativistic matter scaling occurs around the deceleration-to-acceleration transition, where  $\kappa\epsilon_m \sim \alpha$ . The result (58) is a numerical estimation of this deviation. Since  $n$  is very close to 3 at this transition, and the difference  $n - 3$  is negligible in the early universe [30], we may say that  $n \sim 3$  for the entire matter era. Therefore, the deviation of the growth of the

cosmic scale factor in this era from the standard law  $a(t) \sim t^{2/3}$  is very small which is consistent with WMAP cosmological data [55]. The values  $n = 2.93$  and  $n = 2.96$  correspond, respectively, to  $\epsilon = 0.07$  and  $\epsilon = 0.04$ , in agreement with the observational constraint  $\epsilon < 0.1$  [46]. Our results show that the  $R - 1/R$  gravity in the Palatini variational formalism is a viable theory of gravitation that explains current cosmic acceleration. On the other hand, the compatibility of metric  $f(R)$  gravity models with astronomical observations is still an open problem [57].

Finally, we compare the  $R - 1/R$  gravity with the general observational constraint on the matter-dark energy interaction, found in [40]. This constraint is given by

$$r + \frac{\dot{r}}{H(1+r)} < \gamma < -\frac{3w_\Lambda r}{1+r}. \quad (59)$$

However, it is easier to use the conditions from which Eq. (59) was derived. The first one,  $q_0 < 0$  (acceleration), is satisfied by the model (43) which predicts  $q_0 = -0.67_{-0.03}^{+0.06}$  [30]. The second one represents our expectation that the ratio of the matter density to the dark energy density decreases with the evolution of the universe,  $\dot{r} < 0$ . In  $f(R)$  gravity, this condition yields

$$\dot{\tilde{R}} \frac{d}{d\tilde{R}} \left[ \frac{\tilde{R}f' - 3f}{f'(\tilde{R}f' - 2f)} \right] > 0, \quad (60)$$

where  $\dot{\tilde{R}}$  is given by (19). For the case (43), the quantity  $\dot{\tilde{R}}$  is positive in the range of acceleration  $[\tilde{R}_t, \tilde{R}_\infty] = [-\sqrt{5/3}\alpha, -\alpha]$ . Therefore, we obtain

$$\frac{d}{d\tilde{R}} \left[ \frac{\tilde{R} - \frac{2\alpha^2}{3\tilde{R}}}{(1 + \frac{\alpha^2}{3\tilde{R}^2})(\tilde{R} - \frac{\alpha^2}{\tilde{R}})} \right] > 0, \quad (61)$$

which holds so long as  $\tilde{R} < -\alpha/3$ . The last inequality is satisfied for both the matter and dark energy eras of the universe expansion [19,30], and so is the condition  $\dot{r} < 0$ .

## V. SUMMARY

$f(R)$  gravity provides a relativistically covariant explanation for a cosmological constant that varies with time and interacts with matter, basing on the principle of least action. We analyzed the interaction between matter and dark energy in  $f(R)$  gravity formulated in the Einstein conformal frame. We used the Palatini variational principle to obtain the field equations and apply them to a flat, homogeneous, and isotropic universe filled with dust. We found how the simplest phenomenological models of a variable cosmological constant are related to  $f(R)$  gravity. Particularly,  $f(R)$  gravity predicts, for a flat universe without pressure, a simple relation  $\Lambda c^2 = H^2(1 - 2q)$ .

For the particular case  $f(R) = R - \alpha^2/(3R)$ , we found that the interaction rate changes significantly between the moment of the deceleration-to-acceleration transition and now. During the same period, the cosmological constant

does not change much, indicating that its significant decrease must have happened earlier. The predicted value of the current interaction rate is on the order of 1% of the present value of the Hubble parameter, which means that this interaction is relatively weak. Consequently, the energy density scaling only slightly (by  $\sim 1\%$ ) deviates from the standard scaling for nonrelativistic matter, which is consistent with cosmological data. Therefore,  $f(R)$  gravity

in the Palatini formalism appears as a viable theory that explains current cosmic acceleration.

All our predictions for nondimensional cosmological quantities in the  $f(R) = R - \alpha^2/(3R)$  gravity are independent of the value of the only parameter in this model,  $\alpha$ . This is not true for more complicated cases where a function  $f(R)$  contains two or more parameters from which one can compose one or more nondimensional combinations.

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