Is $1 + 2 + 3 + \ldots = -1/12$?

R. Tully-Doyle

Department of Mathematics and Physics
University of New Haven

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Numberphile

Popular (relative to the fact that it's about math) YouTube channel Numberphile ran a video in 2014 that purported to prove the following:

$$1 + 2 + 3 + 4 + \ldots = \frac{-1}{12}.$$

Numberphile's argument

- Define $T_1 = 1 - 1 + 1 - 1 + \dots$. Say that $T_1 = 1/2$.
- Define $T_2 = 1 - 2 + 3 - 4 + 5 \dots$. Then

$$\begin{aligned} 2T_2 &= (1 - 2 + 3 - 4 + \dots) + (1 - 2 + 3 - 4 + \dots) \\ &= 1 + (-2 + 1) + (3 - 2) + (-4 + 3) + \dots \\ &= 1 - 1 + 1 - 1 = T_1 = 1/2. \end{aligned}$$

  and so $T_2 = 1/4$.
- Define $T = 1 + 2 + 3 + \dots$. Then

$$\begin{aligned} T - T_2 &= (1 + 2 + 3 + 4 + \dots) - (1 - 2 + 3 - 4 + \dots) \\ &= (1 - 1) + (2 - -2) + (3 - 3) + (4 - -4) + \dots \\ &= 4 + 8 + 12 + \dots = 4(1 + 2 + 3 + \dots) = 4T. \end{aligned}$$

  Then since $T - T_2 = 4T$ and $T_2 = 1/4$, $T = -1/12$.

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- More lazily: $1 = \sum_{k=1}^{\infty} 2^{-k}$.

Assigning numbers to divergent series

We can think about summation as a function that takes in an infinite series and assigns it a number:

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Is there a summation function that will assign numbers to divergent series? (Like $1-1+1-1\ldots$ and $1-2+3-4+\ldots$.)

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- The define Cesàro summation by

$$\mathrm{CSum}(S) = \lim_{n \to \infty} C_n.$$

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4. So

$$\mathrm{CSum}(1 - 1 + 1 - \ldots) = \lim_{n\to\infty} C_n = \frac{1}{2}.$$

Important properties:

1. If $\text{Sum}(S) = a$ then $\text{CSum}(S) = a$. It is an amusing exercise of real analysis to prove this.
2. Cesàro summation allows a large family of divergent series to be assigned values.
3. CSum is linear: if $\text{CSum}(A)$ and $\text{CSum}(B)$ exist then $\text{CSum}(A) + \text{CSum}(B) = \text{CSum}(A + B)$.

This doesn't fix the Numberphile argument! $T_1 = 1 - 1 + 1 - \dots$ is Cesàro summable but $T_2 = 1 - 2 + 3 - 4 + \dots$ isn't, and neither is $T = 1 + 2 + 3 + \dots$

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Theorem (p-series version 1)

The infinite series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges for $p > 1$. The series diverges for $p \leq 1$.

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Notice that $T = 1 + 2 + 3 + \ldots$ is a *p*-series with $p = -1$.

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Is there a way to assign values to divergent *p*-series (since Cesàro summation isn't going to work)?

A functional approach

Definition (p-series version 2: the Zeta function)

Let $\zeta(p) : (1, \infty) \to \mathbb{R}$ be the function

$$\zeta(p) = \sum_{n=1}^{\infty} \frac{1}{n^p}.$$

ζ is well-defined, since the output of ζ is a convergent p-series for all $p \in (1, \infty)$.

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Exercise: Go read his argument, which is fascinating (and requires an enormous leap of faith that took more than a century before K. Weierstrass was able to make it rigorous).

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$$\zeta(p) = \sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{\Gamma(p)} \int_0^{\infty} \frac{x^{p-1}}{e^x - 1}\, dx.$$

where $\Gamma(p) = \int_0^{\infty} x^{p-1} e^x \, dx$.

Γ is the function used to recover the factorials: $\Gamma(n) = (n-1)!$ when $n \in \mathbb{N}$.

Exercise: Show this. Then show that the integral equation above gives the p-series for a natural number p.

Extension to complex numbers

Once complex analysis was invented, the zeta function got even more awesome.

Theorem (Chebyshev 1840s)

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{x^{s-1}}{e^x - 1}\, dx$$

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Theorem (Riemann 1859)

For all $s \in \mathbb{C} \neq 1$,

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s)\zeta(1-s).$$

Zeta summation

So here we are: for a p-series with $p \neq 1$, define zeta summation by

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For example,

$$\text{ZSum}(\sum \frac{1}{n^2}) = \zeta(2) = \frac{\pi^2}{6}.$$

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- "Divergent series are an invention of the devil..." – N. H. Abel.
- "In mathematics, you don't understand things. You just get used to them." – J. von Neumann.