Escaping non-tangential approach

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Let $D$ and $\Omega$ be proper open subsets of $\mathbb{C}$, and $\partial D$ and $\partial \Omega$ denote their respective topological boundaries. There are a number of classically important families of analytic functions $\varphi : D \to \Omega$ that in addition take the boundary $\partial D$ to the boundary $\partial \Omega$ where defined.

1. Schur functions: $D = \Omega = \mathbb{D}$,
2. Herglotz functions: $D = \mathbb{D}$, $\Omega = \{z : \text{Re } z > 0\}$,
3. Nevanlinna functions: $D = \Omega = \{\text{Re } z > 0\}$.
4. Pick functions $D = \Omega = \{z : \text{Im } z > 0\}$. 
Let $\mathcal{F}$ denote a family of analytic functions $f : D \to \Omega$. While $f$ is analytic on $D$, there is no reason to presume that $f$ is necessarily reasonably behaved on $\partial D$.

We can pose the following question:

**Question**

*Suppose that $f \in \mathcal{F}$ and that $\tau \in \partial D$. Is $f$ “nice” near $\tau$?*
What sorts of substitutions for “nice” are typical?

1. Does $f$ have a limit as $z \to \tau$?
2. Does $f$ have a derivative as $z \to \tau$?
3. Does $f$ have an expansion to order $n$ at $\tau$?

Generally, we call these sorts of conditions boundary regularity.
The Julia quotient $J_f^\tau(z)$ is a function that measures the growth of a function $f$ as the input values $z$ tend to a boundary value $\tau \in \partial D$:

$$J_f^\tau(z) = \frac{\text{dist}(f(z), \partial \Omega)}{\text{dist}(z, \tau)}.$$ 

The Julia quotient is the jumping off point for a theory that understands boundary regularity in terms of restrictions on growth.
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More generally, a Stolz region in a domain $D$ at a point $\tau \in \partial D$ with aperture $M$, denoted $S_{\tau,M}$, to be the set

$$S_{\tau,M} = \{ z \in D \mid \text{dist}(z, \partial D) \geq M \text{dist}(z, \tau) \}.$$
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S_{\tau,M} = \{z \in D | \text{dist}(z, \partial D) \geq M \text{dist}(z, \tau)\}.
\]
Suppose that $D = \Omega = \mathbb{D}$ (that is, $f$ is a Schur function).

**Theorem (Julia-Carathéodory)**

Let $\varphi : \mathbb{D} \to \overline{\mathbb{D}}$ be an analytic function. Let $\tau$ be a point in $T = \partial \mathbb{D}$.

There exist $\omega, \eta \in \partial \mathbb{D}$ so that

$$\varphi = \omega + \eta (t - \tau) + o(|t - \tau|)$$

if and only if the Julia quotient $J^\tau_f$ is bounded on some Stolz region $S_{\tau, M}$.  

The J-C question

One can formulate for a more general notion of regularity what might be called the Julia-Carathédory question:

**Question**

*Is regularity of $f$ at $\tau \in \partial D$ equivalent to boundedness of some growth condition of $f$ at $\tau$?*

This question has been answered in the affirmative in quite a number of more general settings. (More variables, operator valued functions, noncommuting variables, different domains, ...
We move from the disk $\mathbb{D}$ to the upper halfplane $\Pi$: that is, from the Schur functions of Julia-Carathéodory to the class of Pick functions $f : \Pi \to \overline{\Pi}$. 

Theorem (Nevanlinna Representation)

Let $f : \Pi \to \mathbb{C}$. The function $f$ is analytic and maps $\Pi$ to $\Pi$ if and only if there exist $a \in \mathbb{R}$, $b \geq 0$ and a finite positive Borel measure $\mu$ on $\mathbb{R}$ such that

$$f(z) = a + bz + \int_{\mathbb{R}} \frac{1 + tz}{t - z} d\mu(t) \quad (0.1)$$

for all $z \in \Pi$. 
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(0.1)
There is a Corollary that relates boundary regularity to properties of the measure $\mu$.

**Corollary**

An analytic function $f : \Pi \to \overline{\mathbb{P}}$ with Nevanlinna representation

$$f(z) = a + bz + \int_{\mathbb{R}} \frac{1 + tz}{t - z} d\mu(t)$$

is regular to order $2n - 1$ at $\tau$ if and only if $\frac{1}{(t - \tau)^{2n}}$ is integrable with respect to $\mu$. 
Let \( \gamma : [0, \infty) \rightarrow \mathbb{R}^{\geq 0} \) be a monotone increasing function such that \( \gamma(t) \) is \( O(t^2) \).

\( f : \Pi \rightarrow \overline{\Pi} \) is \( \gamma \)-regular whenever there exists a \( C > 0 \) such that

\[
\frac{1}{\gamma(C|t|)}
\]

is integrable with respect to the \( \mu \).
The big idea

Let $\gamma : [0, \infty) \rightarrow \mathbb{R}^{\geq 0}$ be a monotone increasing function such that $\gamma(t)$ is $O(t^2)$.

$f : \Pi \rightarrow \overline{\Pi}$ is $\gamma$-regular whenever there exists a $C > 0$ such that $\frac{1}{\gamma(C|t|)}$ is integrable with respect to the $\mu$.

For example, when $\gamma(t) = (t - \tau)^{2n}$, we recover the previous corollary: $f$ is $\gamma$-regular iff $f$ is regular to order $2n - 1$. 
An illustrative example

Suppose that $\gamma(t) = e^{-1/t}$.
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It turns out that $\gamma$-regularity in this case implies \textbf{analytic determinacy} of the measure $\mu$: that is the values of the moments of $\mu$ uniquely determine the measure.
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In a \( \lambda\text{-Stolz region} \), the rays at \( \tau \) in a standard Stolz region are replaced by a curve \( \lambda \).
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In a $\lambda$-Stolz region, the rays at $\tau$ in a standard Stolz region are replaced by a curve $\lambda$.

If $\lambda(t) = \frac{e^{-1/t}}{t}$, then the associated $\lambda$-Stolz region at $\tau = 0$ looks like: picture here
Amortized Julia quotient

Let $S_0^\lambda$ be a $\lambda$-Stolz region. Let $d > 0$, and let $C_d$ denote the set of points in $\Pi$ that are distance $\lambda(d)$ from the boundary $\mathbb{R}$.

The amortized Julia quotient of $f$ with respect to $\lambda$ at $\tau$, denoted $AJ^\tau_{\varphi,\lambda}(d)$, is defined as

$$AJ^\tau_{f,\lambda}(d) = \frac{1}{|C_d|} \int_{C_d} J^\tau_{f}(z) \, d|z|,$$  (0.2)

where $|C_d|$ denotes arclength.
A representative $\gamma$ J-C theorem

Theorem (Pascoe, Seargent, T.D.)

Let $f$ be an analytic function from $\Pi \to \bar{\Pi}$ and suppose that
$\gamma(t) = e^{-1/t}$ and that $\lambda(t) = \frac{e^{-1/t}}{t}$.

$f$ is $\gamma$-regular if and only if $AJ_f^0(z)$ is bounded on $S_0^{\lambda(Ct)}$ for some $C > 0$. 
Theorem (Pascoe, Seargent, T.D.)

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$f$ is $\gamma$-regular if and only if $AJ_f(z)$ is bounded on $S_0^{\lambda(Ct)}$ for some $C > 0$ if and only if $J_f(z)$ is bounded on $S_0^{\lambda(Ct)}$ for some $C > 0$. 
Extensions and open questions

1. $\gamma$-regularity recovers Bolotnikov-Kheifits higher order boundary differentiability.

2. We only have partial results on the disk: in the case that we can use $J$ instead of $AJ$, can Cayley transform back.

3. Moment determinacy in free probability?

4. Two variables?