Escaping non-tangential approach

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Boundary behavior

Let D and Ω be proper open subsets of $\mathbb{C}$, and ∂D and $\partial\Omega$ denote their respective topologial boundaries. There are a number of classically important families of analytic functions $\varphi : D \to \Omega$ that in addition take the boundary ∂D to the boundary $\partial\Omega$ where defined.

1. Schur functions: $D = \Omega = \mathbb{D}$,
2. Herglotz functions: $D = \mathbb{D}$, $\Omega = \{z : \mathrm{Re}\ z > 0\}$,
3. Nevanlinna functions: $D = \Omega = \{\mathrm{Re}\ z > 0\}$.
4. Pick functions $D = \Omega = \{z : \mathrm{Im}\ z > 0\}$.

Let $\mathcal{F}$ denote a family of analytic functions $f : D \to \Omega$. While f is analytic on D, there is no reason to presume that f is necessarily reasonably behaved on ∂D.

We can pose the following question:

Question

Suppose that $f \in \mathcal{F}$ and that $\tau \in \partial D$. Is f "nice" near τ?

What sorts of substitutions for "nice" are typical?

1. Does f have a limit as $z \to \tau$?
2. Does f have a derivative as $z \to \tau$?
3. Does f have an expansion to order n at τ?

Generally, we call these sorts of conditions boundary regularity.

The Julia quotient $J_f^\tau(z)$ is a function that measures the growth of a function f as the input values z tend to a boundary value $\tau \in \partial D$:

$$J_f^\tau(z) = \frac{\text{dist}(f(z), \partial\Omega)}{\text{dist}(z, \tau)}.$$

The Julia quotient is the jumping off point for a theory that understands boundary regularity in terms of restrictions on growth.

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More generally, a Stolz region in a domain D at a point $\tau \in \partial D$ with aperture M, denoted $S_{\tau,M}$, to be the set

$$S_{\tau,M} = \{z \in D | \operatorname{dist}(z, \partial D) \geq M \operatorname{dist}(z, \tau)\}.$$

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Classical J-C Theorem on $\mathbb{D}$

Suppose that $D = \Omega = \mathbb{D}$ (that is, f is a Schur function).

Theorem (Julia-Carathéodory)

Let $\varphi : \mathbb{D} \to \overline{\mathbb{D}}$ be an analytic function. Let τ be a point in $\mathbb{T} = \partial\mathbb{D}$.

There exist $\omega, \eta \in \partial\mathbb{D}$ so that

$$\varphi = \omega + \eta(t - \tau) + o(|t - \tau|)$$

if and only if the Julia quotient J_f^τ is bounded on some Stolz region $S_{\tau,M}$.

The J-C question

One can formulate for a more general notion of regularity what might be called the Julia-Carathódory question:

Question

Is regularity of f at $\tau \in \partial D$ equivalent to boundedness of some growth condition of f at τ?

This question has been answered in the affirmative in quite a number of more general settings. (More variables, operator valued functions, noncommuting variables, different domains, ...)

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Theorem (Nevanlinna Representation)

Let $f : \Pi \to \mathbb{C}$. The function f is analytic and maps Π to $\overline{\Pi}$ if and only if there exist $a \in \mathbb{R}$, $b \geq 0$ and a finite positive Borel measure μ on $\mathbb{R}$ such that

$$f(z) = a + bz + \int_{\mathbb{R}} \frac{1 + tz}{t - z} \mathrm{d}\mu(t) \tag{0.1}$$

for all $z \in \Pi$.

There is a Corollary that relates boundary regularity to properties of the measure μ.

Corollary

An analytic function $f : \Pi \to \overline{\Pi}$ with Nevanlinna representation

$$f(z) = a + bz + \int_{\mathbb{R}} \frac{1 + tz}{t - z} \mathrm{d}\mu(t)$$

is regular to order $2n - 1$ at τ if and only if $\frac{1}{(t-\tau)^{2n}}$ is integrable with respect to μ.

The big idea

Let $\gamma : [0, \infty) \to \mathbb{R}^{\geq 0}$ be a monotone increasing function such that $\gamma(t)$ is $O(t^2)$.

$f : \Pi \to \overline{\Pi}$ is γ-**regular** whenever there exists a $C > 0$ such that $\frac{1}{\gamma(C|t|)}$ is integrable with respect to the μ.

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For example, when $\gamma(t) = (t - \tau)^{2n}$, we recover the previous corollary: f is γ-regular iff f is regular to order $2n - 1$.

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It turns out that γ-regularity in this case implies **analytic determinacy** of the measure μ: that is the values of the moments of μ uniquely determine the measure.

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In a λ-**Stolz region**, the rays at τ in a standard Stolz region are replaced by a curve λ.
If $\lambda(t) = \frac{e^{-1/t}}{t}$, then the associated λ-Stolz region at $\tau = 0$ looks like: picture here

Amortized Julia quotient

Let S_0^λ be a λ-Stolz region. Let $d > 0$, and let C_d denote the set of points in Π that are distance $\lambda(d)$ from the boundary $\mathbb{R}$.

The **amortized Julia quotient of f with respect to λ at τ, denoted** $AJ^\tau_{\varphi,\lambda}(d)$, is defined as

$$AJ^\tau_{f,\lambda}(d) = \frac{1}{|C_d|} \int_{C_d} J^\tau_f(z)\,\mathrm{d}|z|, \tag{0.2}$$

where $|C_d|$ denotes arclength.

A representative γ J-C theorem

Theorem (Pascoe, Seargent, T.D.)

Let f be an analytic function from $\Pi \to \overline{\Pi}$ and suppose that $\gamma(t) = e^{-1/t}$ and that $\lambda(t) = \frac{e^{-1/t}}{t}$.

f is γ-regular
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Extensions and open questions

1. γ-regularity recovers Bolotnikov-Kheifits higher order boundary differentiability.
2. We only have partial results on the disk: in the case that we can use J instead of AJ, can Cayley transform back.
3. Moment determinacy in free probability?
4. Two variables?