Second Derivative Test for $f : \mathbb{R}^2 \to \mathbb{R}$

Marc Mehlman

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**Theorem 0.1** Let $f : B \subseteq \mathbb{R}^2 \to \mathbb{R}$ and assume in a neighborhood of a critical point, $(x_0, y_0)$, of $f$ that $f_x, f_y, f_{xx}, f_{xy} \& f_{yy}$ exists. Let

$$d \overset{\text{def}}{=} f_{xx}(x_0, y_0)f_{yy}(x_0, y_0) - f_{xy}^2(x_0, y_0).$$

Then

1. $d > 0$ and $f_{xx}(x_0, y_0) > 0$ $\Rightarrow$ *minimum*
2. $d > 0$ and $f_{xx}(x_0, y_0) < 0$ $\Rightarrow$ *maximum*
3. $d < 0$ $\Rightarrow$ *saddle point*
4. $d = 0$ no claim.

**Proof:** $\nabla f(x_0, y_0) = \vec{0}$ since $(x_0, y_0)$ is a critical point. Fix $(u_1, u_2) \in \mathbb{R}^2$ and define $p : [-1, 1] \to \mathbb{R}^2$

$$t \mapsto (x_0, y_0) + t(u_1, u_2)$$

Use the 2nd derivative test on $f \circ p$ to determine type of extrema (for all possible $(u_1, u_2)$).

$$\frac{d^2}{dt^2}(f \circ p)(t) = \frac{d}{dt}[Df(p(t))Dp(t)] \quad \text{chain rule}$$

$$= \frac{d}{dt}[\nabla f(p(t)) \cdot (u_1, u_2)]$$

$$= \frac{d}{dt}[\nabla f \circ p(t) \cdot (u_1, u_2) + \nabla f(p(t)) \cdot (0, 0)]$$

$$= D\nabla f(p(t))Dp(t) \cdot (u_1, u_2) \quad \text{chain rule}$$

$$= \begin{bmatrix}
  f_{xx}(p(t)) & f_{xy}(p(t)) \\
  f_{yx}(p(t)) & f_{yy}(p(t))
\end{bmatrix}
\begin{bmatrix}
  u_1 \\
  u_2
\end{bmatrix}
\cdot
\begin{bmatrix}
  u_1 \\
  u_2
\end{bmatrix}$$

$$= f_{xx}(p(t))u_1^2 + 2f_{xy}(p(t))u_1u_2 + f_{yy}(p(t))u_2^2.$$
Consider the following equation in \((x_0, y_0)\), the level surface \(\frac{d^2}{dt^2}(f \circ p)(0) = K\)

\[
\frac{d^2}{dt^2}(f \circ p)(0) = f_{xx}(x_0, y_0)u_1^2 + 2f_{xy}(x_0, y_0)u_1u_2 + f_{yy}(x_0, y_0)u_2^2 = K.
\]

To figure out what type of conic section this level curve is, consider the discriminant

\[B^2 - 4AC = (2f_{xy}(x_0, y_0))^2 - 4f_{xx}(x_0, y_0)f_{yy}(x_0, y_0) = -4d.\]

Thus if

1. \(d > 0\) the level curves are ellipses and only occur on one side of \(xy\)-plane ⇒ the second derivative of \(f(p(t))\) is always of the same sign at the critical point \((x_0, y_0)\) no matter what \((u_1, u_2)\) is ⇒ there is a max or min (depending on the sign of the second derivative) at \((x_0, y_0)\)

2. \(d < 0\) the level curves are hyperbolas and thus the second derivative of \(f(p(t))\) can be either positive or negative, depending on \((u_1, u_2)\) ⇒ saddle point.

**Example 0.2** Find critical points of \(h(x, y) = x^2 - y^2 - 2x - 4y - 4\) and use the Second Derivative Test to determine if the critical point is located at a maximum, minimum or a saddle point.

**Sol:**

\[
0 = h_x(x, y) = 2x - 2 \\
0 = h_y(x, y) = -2y - 4
\]

Thus \((1, -2)\) is the critical points. Notice that since

\[
f_{xx}(x, y) = 2 \\
f_{xy}(x, y) = 0 \\
f_{yy}(x, y) = -2
\]

one has

\[d = 2(-2) - 0^2 = -4 < 0.\]

Thus \((1, -2)\) is a saddle point.