Mixed Partial Are Equal

**Theorem 0.1 (Clairaut)** $f$ is defined on a disk $D$ containing the point $(a,b)$. If the functions $f_{xy}$ and $f_{yx}$ are both continuous on $D$, then $f_{xy}(a,b) = f_{yx}(a,b)$. □

**Proof** For small values of $h$, $h \neq 0$, consider the difference

$$
\Delta(h) = [f(a+h, b+h) - f(a+h, b)] - [f(a, b+h) - f(a, b)]
$$

Notice that if we let $g(x) = f(x, b+h) - f(x, b)$, then

$$
\Delta(h) = g(a+h) - g(a)
$$

By the Mean Value Theorem, there is a number $c$ between $a$ and $a+h$ such that

$$
g(a+h) - g(a) = g'(c)h = h[f_x(c, b+h) - f_x(c, b)]
$$

Applying the Mean Value Theorem again, this time to $f_x$, we get a number $d$ between $b$ and $b+h$ such that

$$
f_x(c, b+h) - f_x(c, b) = f_{xy}(c, d)h
$$

Combining these equations, we obtain

$$
\Delta(h) = h^2 f_{xy}(c, d)
$$

If $h \to 0$, then $(c, d) \to (a, b)$, so the continuity of $f_{xy}$ gives

$$
\lim_{h \to 0} \frac{\Delta(h)}{h^2} = \lim_{(c,d) \to (a,b)} f_{xy}(c, d) = f_{xy}(a, b)
$$

Similarly by writing

$$
\Delta(h) = [f(a+h, b+h) - f(a, b+h)] - [f(a+h, b) - f(a, b)]
$$

and using the Mean Value Theorem twice and the continuity of $f_{yx}$ at $(a,b)$, we obtain

$$
\lim_{h \to 0} \frac{\Delta(h)}{h^2} = f_{yx}(a, b)
$$

It follows that $f_{xy}(a,b) = f_{yx}(a,b)$. ■