Noncommutative momentum and torsional regularization

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Abstract

We derive the quantum commutation relation for the four-momentum in the presence of torsion and traced over spinor indices. In the Einstein–Cartan theory of gravity, in which the torsion tensor is coupled to the spin of fermions, this relation can be reduced to a commutation relation for the momentum components.

We propose that this relation replaces the integration in the momentum space in Feynman diagrams with the summation over the discrete momentum eigenvalues. We derive a prescription for this summation that agrees with convergent integrals.

We show that this prescription regularizes ultraviolet-divergent integrals in loop diagrams. We extend this prescription to tensor integrals.

We derive a finite, gauge-invariant vacuum polarization tensor and a finite running coupling. Including loops from all charged fermions, we find a finite value for the bare electric charge of an electron. This torsional regularization, originating from the torsion-generated noncommutativity of the momentum, may therefore provide a realistic, physical mechanism for eliminating infinities in quantum field theory and making renormalization finite.

Problems of general relativity

General relativity describes gravity as curvature of spacetime.

- Singularities: points with infinite density of matter.
- Incompatible with quantum mechanics. We need quantum gravity. It may resolve the singularity problem.
- Field equations contain the conservation of orbital angular momentum, contradicting Dirac equation which gives the conservation of total angular momentum (orbital + spin) and allows spin-orbit exchange in QM.

Simplest extension of GR to include QM spin:
Einstein–Cartan theory. It also eliminates the singularity problem.

Einstein-Cartan-Sciama-Kibble gravity

- Spacetime has curvature and **torsion**. $S^k{}_{ij} = \Gamma^{\ \ k}_{[i\,j]}$

- Lagrangian density is proportional to curvature scalar (as in GR).

- Cartan equations:

Torsion is proportional to **spin** density of fermions. ECSK differs significantly from GR at densities > 10^{45} kg/m^3; passes all tests.

$$S_{jik} - S_i g_{jk} + S_k g_{ji} = -\frac{1}{2}\kappa s_{ikj}$$

arXiv.org > gr-qc > arXiv:0911.0334

- Einstein equations:

Curvature is proportional to **energy and momentum** density.

$$G^{ik} = \kappa T^{ik} + \frac{1}{2}\kappa^2\left(s^{ij}{}_j s^{kl}{}_l - s^{ij}{}_l s^{kl}{}_j - s^{ijl}s^k{}_{jl} + \frac{1}{2}s^{jli}s_{jl}{}^k + \frac{1}{4}g^{ik}(2s_j{}^l{}_m s^{jm}{}_l - 2s_j{}^l{}_l s^{jm}{}_m + s^{jlm}s_{jlm})\right)$$

Universe with spin fluid in a black hole

Dirac particles can be averaged macroscopically as a spin fluid.

Einstein-Cartan equations for a homogeneous and isotropic Universe become Friedmann equations for the scale factor *a*.

$$\frac{\dot{a}^2}{c^2} + 1 = \frac{1}{3}\kappa\Big(\epsilon - \frac{1}{4}\kappa s^2\Big)a^2$$
$$\frac{\dot{a}^2 + 2a\ddot{a}}{c^2} + 1 = -\kappa\Big(p - \frac{1}{4}\kappa s^2\Big)a^2$$

$$s^2 = \frac{1}{8}(\hbar c n)^2$$

NP, JETP (2020); arXiv:2008.02136

Spin and torsion modify the energy density and pressure with a negative term proportional to the square of the fermion number density *n*, which acts like repulsive gravity and prevents singularities. The Big Bang is replaced by a non-singular Big Bounce.

Problems of quantum field theory

- Ultraviolet divergence: Feynman diagrams involve divergent integrals in the four-momentum space arising from high-energy contributions.
- This unphysical result requires regularization: a mathematical method of turning singular quantities into finite quantities. Most common: adding fictitious particles, changing dimensions.
- Renormalization: the original (bare) values of mass and charge absorb divergent terms, giving the measured (dressed) values.
- Dirac was critical about renormalization and expected a realistic regularization based on the principles of physics.
- Solution: torsional regularization, renormalization is finite.

Torsion and noncommutativity of momentum

- Consider two infinitesimal four-vectors *dx* and *dx'*.
- In the presence of torsion the parallel transport of *dx* along *dx'* and the parallel transport of *dx'* along *dx* do not form a closed parallelogram:

$$\delta dx^i = -\Gamma^{\,i}_{jk} dx^j dx'^k \qquad \delta dx'^i = -\Gamma^{\,i}_{jk} dx^j dx'^k \qquad \delta dx'^i - \delta dx^i = -S^i{}_{jk} dx^j dx'^k$$

- Since the momentum is a generator of translation, described by the parallel transport, its operator in quantum mechanics is given by the covariant derivative:

$$p_k = i\hbar\nabla_k$$

- In the presence of torsion, translations do not commute and therefore the four-momentum components do not commute:

$$[p_i, p_j] = 2i\hbar S^k{}_{ij} p_k$$

Integration in momentum space becomes summation over momentum eigenstates

- The classical and quantum partition functions in statistical physics:

$$\int dq \int dp\, f(H(q,p)) \leftrightarrow 2\pi \sum_{\text{eigenstates}} f(E)\, |[q,p]|$$

- One can choose locally a frame of reference in which only the space momentum components do not commute:

$$[p_x, p_y] = iQp_z, \quad [p_y, p_z] = iQp_x, \quad [p_z, p_x] = iQp_y \qquad Q = -2\hbar A^0 \qquad A^i = \frac{1}{6}\epsilon^{ijkl} S_{jkl}$$

- Einstein-Cartan gravity gives: $Q = Up^3$ (U is const ~ M^{-2}_{Pl})

- We obtain a relation analogous to the angular momentum:

$$[n_x, n_y] = in_z, \quad [n_y, n_z] = in_x, \quad [n_z, n_x] = in_y \qquad \mathbf{n} = \frac{\mathbf{p}}{Q}$$

Integration in momentum space becomes summation over momentum eigenstates

- We propose that the integration in *n*-space satisfying

$$[n_x, n_y] = i n_z, \quad [n_y, n_z] = i n_x, \quad [n_z, n_x] = i n_y$$

- Is replaced with the summation:

$$\int dn_x \int dn_y \int dn_z \, f(\mathbf{n}^2) \to 4\pi \sum_{\text{eigenstates}} f(\mathbf{n}^2)\,|n_z|$$

$$\to 4\pi \sum_{l=1}^{\infty} \sum_{m=-l}^{l} f(\mathbf{n}^2)\,|m| = 4\pi \sum_{l=1}^{\infty} f(\mathbf{n}^2)\, l(l+1)$$

- If the integral is finite, the corresponding sum is almost equal.

- Torsional regularization: NP, Found. Phys. 50, 900 (2020)

Integration in momentum space becomes summation over momentum eigenstates

- Apply TR to a logarithmically divergent integral:

$$\int \frac{d^4p}{(p^2+\mu^2)^2} = \int \frac{dp_0\, d\mathbf{p}}{(p^2+\mu^2)^2} = \int \frac{dp_0\, J\, d\mathbf{n}}{(p^2+\mu^2)^2} \to 4\pi \int_{-\infty}^{\infty} dp_0 \sum_{l=1}^{\infty} \frac{J}{(p^2+\mu^2)^2}\, l(l+1)$$

$$J = \partial(p_x,p_y,p_z)/\partial(n_x,n_y,n_z)$$

$$p^2 = p_0^2 + U^2 n^2 p^6$$

$$\frac{\partial p}{\partial n_x} = \frac{U^2 p^5 n_x}{1-3U^2n^2p^4}$$

$$\frac{\partial p_x}{\partial n_x} = \frac{\partial(Qn_x)}{\partial n_x} = Q + 3Un_xp^2\frac{\partial p}{\partial n_x}$$

$$\frac{\partial p_x}{\partial n_y} = \frac{\partial(Qn_x)}{\partial n_y} = 3Un_xp^2\frac{\partial p}{\partial n_y}$$

$$dp_0/dp = (1-3U^2n^2p^4)/(1-U^2n^2p^4)^{1/2}$$

$$n = \sqrt{l(l+1)}$$

$$J = \det\begin{pmatrix} \partial p_x/\partial n_x & \partial p_x/\partial n_y & \partial p_x/\partial n_z \\ \partial p_y/\partial n_x & \partial p_y/\partial n_y & \partial p_y/\partial n_z \\ \partial p_z/\partial n_x & \partial p_z/\partial n_y & \partial p_z/\partial n_z \end{pmatrix} = \frac{Q^3}{1-3U^2n^2p^4}$$

Torsion eliminates ultraviolet divergence

$$4\pi\int_{-\infty}^{\infty}dp_0\sum_{l=1}^{\infty}\frac{Q^3n^2}{(1-3U^2n^2p^4)(p^2+\mu^2)^2}=4\pi\int dp\frac{dp_0}{dp}\sum_{l=1}^{\infty}\frac{Q^3n^2}{(1-3U^2n^2p^4)(p^2+\mu^2)^2}$$

$$=4\pi\int_{-1/\sqrt{Un}}^{1/\sqrt{Un}}dp\sum_{l=1}^{\infty}\frac{Q^3n^2}{(1-U^2n^2p^4)^{1/2}(p^2+\mu^2)^2}=8\pi\int_0^{1/\sqrt{Un}}dp\sum_{l=1}^{\infty}\frac{U^3p^9n^2}{(1-U^2n^2p^4)^{1/2}(p^2+\mu^2)^2}$$

$$=8\pi\int_0^1 d\xi\sum_{l=1}^{\infty}\frac{U^3\xi^9n^2(Un)^{-5}}{(1-\xi^4)^{1/2}[\xi^2/(Un)+\mu^2]^2}=8\pi\int_0^1 d\xi\sum_{l=1}^{\infty}\frac{\xi^9n^{-1}}{(1-\xi^4)^{1/2}[\xi^2+U\mu^2n]^2}$$

$$=4\pi\int_0^1 d\zeta\sum_{l=1}^{\infty}\frac{\zeta^4n^{-1}}{(1-\zeta^2)^{1/2}[\zeta+U\mu^2n]^2}=4\pi\sum_{l=1}^{\infty}\int_0^{\pi/2}d\phi\frac{\sin^4\phi\, n^{-1}}{[\sin\phi+U\mu^2n]^2}$$

$$=4\pi\sum_{l=1}^{\infty}\int_0^{\pi/2}d\phi\frac{\sin^4\phi\,[l(l+1)]^{-1/2}}{[\sin\phi+U\mu^2\sqrt{l(l+1)}]^2},\qquad Unp^2=\xi^2=\zeta=\sin\phi$$

The logarithmically divergent integral is replaced with a sum that converges as l^{-3}.

The convergence follows from the separation between the momentum eigenvalues that increases with magnitude.

Torsion eliminates ultraviolet divergence

This procedure can be generalized to tensor integrals:

$$\int \frac{d^4p}{(p^2+\mu^2)^s} \to 8\pi \int_0^{1/\sqrt{Un}} dp \sum_{l=1}^{\infty} \frac{U^3 p^9 n^2}{(1-U^2n^2p^4)^{1/2}(p^2+\mu^2)^s}$$
$$= 8\pi \int_0^1 d\xi \sum_{l=1}^{\infty} \frac{U^3\xi^9 n^2 (Un)^{-5}}{(1-\xi^4)^{1/2}[\xi^2/(Un)+\mu^2]^s} = 8\pi \int_0^1 d\xi \sum_{l=1}^{\infty} \frac{U^{s-2}\xi^9 n^{s-3}}{(1-\xi^4)^{1/2}[\xi^2+U\mu^2 n]^s}$$
$$= 4\pi \int_0^1 d\zeta \sum_{l=1}^{\infty} \frac{U^{s-2}\zeta^4 n^{s-3}}{(1-\zeta^2)^{1/2}[\zeta+U\mu^2 n]^s} = 4\pi U^{s-2} \sum_{l=1}^{\infty} \int_0^{\pi/2} d\phi \frac{\sin^4\phi\, n^{s-3}}{[\sin\phi + U\mu^2 n]^s}$$
$$= 4\pi U^{s-2} \sum_{l=1}^{\infty} \int_0^{\pi/2} d\phi \frac{\sin^4\phi\,[l(l+1)]^{(s-3)/2}}{[\sin\phi + U\mu^2\sqrt{l(l+1)}]^s}.$$

$$\int d^4p \frac{\partial}{\partial p_\nu}\Big(\frac{p^\mu}{(p^2+\Delta)^s}\Big) = \int d^4p \frac{\delta^{\mu\nu}}{(p^2+\Delta)^s} - 2s\int d^4p \frac{p^\mu p^\nu}{(p^2+\Delta)^{s+1}}$$

$$\int d^4p \frac{p^\mu p^\nu}{(p^2+\Delta)^s} = \frac{\delta^{\mu\nu}}{2(s-1)} \int d^4p \frac{1}{(p^2+\Delta)^{s-1}}$$

NP, Found. Phys. 50, 900 (2020)

Vacuum polarization

The vacuum polarization tensor is gauge invariant:

$$\Pi^{\mu\nu}_{\text{bubble}}(q) = -\frac{\alpha_0}{\pi^3}\int d^4p_{\text{E}}\int_0^1 dx\frac{-2p^\mu_{\text{E}}p^\nu_{\text{E}} + p^2_{\text{E}}\delta^{\mu\nu} + \Delta\delta^{\mu\nu} + 2(q^2g^{\mu\nu} - q^\mu q^\nu)x(1-x)}{(p^2_{\text{E}} + \Delta)^2}$$
$$= -\frac{2\alpha_0}{\pi^3}\int d^4p_{\text{E}}\int_0^1 dx\frac{x(1-x)}{(p^2_{\text{E}}+\Delta)^2}(q^2g^{\mu\nu} - q^\mu q^\nu) = \Pi(q^2)q^2\left(g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}\right),$$

$$\Pi(q^2) = -\frac{2\alpha_0}{\pi^3}\int d^4p_{\text{E}}\int_0^1 dx\frac{x(1-x)}{(p^2_{\text{E}}+\Delta)^2}$$

$$\Delta = m^2 - q^2x(1-x)$$

$$\Pi(q^2) \to -\frac{8\alpha_0}{\pi^2}\sum_{l=1}^{\infty}\int_0^1 dx\int_0^{\pi/2} d\phi\frac{\sin^4\phi\, n^{-1}x(1-x)}{[\sin\phi + U\Delta n]^2}$$
$$= -\frac{8\alpha_0}{\pi^2}\sum_{l=1}^{\infty}\int_0^1 dx\int_0^{\pi/2} d\phi\frac{\sin^4\phi\,[l(l+1)]^{-1/2}x(1-x)}{[\sin\phi + U\Delta\sqrt{l(l+1)}]^2}$$

The sum-integral in Π is finite.

Torsion makes bare charge finite

Renormalization of the electric charge:

$$\alpha = \frac{\alpha_0}{1 - \Pi(0)} \qquad \alpha_{\rm run} = \frac{\alpha_0}{1 - \Pi(q^2)}$$

Gives the bare electric charge of an electron:

$$e_0 = \frac{e}{(1 + \Pi_{\rm R}(0))^{1/2}} = e\Big[1 - \frac{8\alpha}{\pi^2}\sum_{l=1}^{\infty}\int_0^1 dx \int_0^{\pi/2} d\phi \frac{\sin^4\phi\, n^{-1} x(1-x)}{[\sin\phi + U m^2 n]^2}\Big]^{-1/2}$$

Including all charged fermions in *Π* gives the bare charge -1.22 e.
The running coupling constant is finite.

Accordingly, the bare fine structure constant is about 1/92.1.
NP, Found. Phys. 50, 900 (2020); arXiv:1712.09997

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Summary

- The conservation law for total angular momentum (orbital + spin) in curved spacetime, consistent with Dirac equation, requires torsion.
- In the presence of torsion, the four-momentum operator components do not commute. The integration in the momentum space must be replaced with the summation over the momentum eigenvalues.
- The separation between the momentum eigenvalues increases with the magnitude of the momentum as a result of the Einstein–Cartan gravity. Consequently, ultraviolet divergent integrals turn into convergent sums.
- Torsion naturally regularizes ultraviolet divergence in QED. Renormalization in QED is finite, leading to a finite bare charge of an electron: -1.22 e.
- Future work: research how torsion affects the electroweak and strong interactions.