# General-Relativistic Pilot-Wave Quantum Mechanics

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12th Australasian Conference
on General Relativity and Gravitation

University of Tasmania, Hobart, Australia

November 28, 2023

# Affine connection & torsion

The **affine connection** $\Gamma^k_{ij}$ allows to construct the covariant derivative $\nabla_i$ of a vector that transforms under coordinate transformations like a tensor:

$$\nabla_i V^k = \partial_i V^k + \Gamma^k_{ji} V^j, \quad \nabla_i V_k = \partial_i V_k - \Gamma^j_{ki} V_j.$$

For tensors with more indices, each index produces a term with the affine connection. The affine connection is not a tensor, but its variation is a tensor. The antisymmetric part of the affine connection defines the **torsion tensor**:

$$S^k{}_{ij} = \frac{1}{2}(\Gamma^k_{ij} - \Gamma^k_{ji}). \tag{1}$$

The curvature tensor is given by $R^i{}_{klm} = \partial_l \Gamma^i_{km} - \partial_m \Gamma^i_{kl} + \Gamma^j_{km} \Gamma^i_{jl} - \Gamma^j_{kl} \Gamma^i_{jm}$.

The covariant derivative of the metric tensor is zero (metricity): $\nabla_j g_{ik} = 0$. This relation determines the affine connection in terms of the metric, its partial derivatives, and torsion:

$$\Gamma^k_{ij} = \mathring{\Gamma}^k_{ij} + S^k{}_{ij} + S_{ij}{}^k + S_{ji}{}^k,$$

where the Christoffel symbols (Levi-Civita connection) are given by

$$\mathring{\Gamma}^j_{ik} = \frac{1}{2} g^{jl} (\partial_i g_{kl} + \partial_k g_{il} - \partial_l g_{ik}).$$

# Dirac equation & tetrad

In a flat spacetime, the Dirac equation for a free particle with mass $m$ is given by

$$i\hbar\gamma^\mu\partial_\mu\psi = mc\psi,$$

where $\psi$ is the four-component wave function and $\gamma^\mu$ are the $4\times4$ Dirac matrices, satisfying the relation

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu}I_4,$$

where $\eta^{\mu\nu} = \text{diag}(1,-1,-1,-1)$ is the Minkowski metric tensor and $I_4$ is the four-dimensional unit matrix. In the presence of the electromagnetic field, $\partial_\mu$ is extended to $\partial_\mu + iqA_\mu$, where $A_\mu$ is the electromagnetic potential and $q$ is the electric charge of the particle.

In a curved spacetime, the partial derivatives $\partial_\mu$ must be replaced with the covariant derivatives. At every point in spacetime, in addition to a general coordinate system, it is possible to set up a **tetrad**: four linearly independent vectors $e^i_\mu$ such that

$$e^i_\mu e^k_\nu g_{ik} = \eta_{\mu\nu}.$$

The tetrad relates the Greek vector indices of the locally flat, Lorentz coordinate system (of special relativity) to the Latin vector indices of the general coordinate system (of general relativity): $V^i = e^i_\mu V^\mu$. The choice of the tetrad is not unique: a **Lorentz transformation** $\tilde{e}^i_\mu = \Lambda^\nu_\mu e^i_\nu$, where $\Lambda^\nu_\mu$ are the Lorentz matrices satisfying $\Lambda^\rho_\mu\Lambda^\sigma_\nu\eta_{\rho\sigma} = \eta_{\mu\nu}$, produces a new tetrad $\tilde{e}^i_\mu$.

# Spin connection

The **spin connection** is given by

$$\omega^{\mu}{}_{\nu i} = e^{\mu}_{k}\omega^{k}{}_{\nu i} = e^{\mu}_{k}\nabla_{i}e^{k}_{\nu} = e^{\mu}_{k}(\partial_{i}e^{k}_{\nu} + \Gamma^{k}_{ji}e^{j}_{\nu}). \tag{2}$$

The spin connection allows to extend covariant differentiation to vectors with Lorentz indices:

$$\nabla_{i}V^{\mu} = \partial_{i}V^{\mu} + \omega^{\mu}{}_{\nu i}V^{\nu}, \quad \nabla_{i}V_{\mu} = \partial_{i}V_{\mu} - \omega^{\nu}{}_{\mu i}V_{\nu}.$$

For tensors with Lorentz indices, each index produces a term with the spin connection. Consequently, the covariant derivative of a tetrad is zero:

$$\nabla_{k}e^{i}_{\mu} = \partial_{k}e^{i}_{\mu} + \Gamma^{i}_{jk}e^{j}_{\mu} - \omega^{\nu}{}_{\mu k}e^{i}_{\nu} = 0.$$

Therefore, the covariant differentiation commutes with converting between coordinate and Lorentz indices. This relation determines the spin connection in terms of the affine connection, the tetrad, and its partial derivatives.

The metricity of the affine connection leads to $\nabla_{j}g_{ik} = e^{\mu}_{i}e^{\nu}_{k}\nabla_{j}\eta_{\mu\nu}$ $= -e^{\mu}_{i}e^{\nu}_{k}(\omega^{\rho}{}_{\mu j}\eta_{\rho\nu} + \omega^{\rho}{}_{\nu j}\eta_{\mu\rho}) = -(\omega_{kij} + \omega_{ikj}) = 0$. Consequently, $\nabla_{i}\eta_{\mu\nu} = 0$ and the spin connection is antisymmetric in its first two indices: $\omega_{\mu\nu i} = -\omega_{\nu\mu i}$.

# Spinor representation of Lorentz group

Let $L$ be a 4×4 matrix such that

$$\gamma^\mu = \Lambda^\mu_\nu L \gamma^\nu L^{-1}.$$

This condition represents the constancy of the Dirac matrices under a Lorentz tetrad rotation combined with a similarity transformation and gives the matrix $L$ as a function of the Lorentz matrix $\Lambda^\mu_\nu$. For an infinitesimal Lorentz transformation $\Lambda^\mu_\nu = \delta^\mu_\nu + \epsilon^\mu{}_\nu$, where $\epsilon_{\mu\nu} = -\epsilon_{\nu\mu}$ are infinitesimal quantities, the solution for $L$ is

$$L = I_4 + \frac{1}{2}\epsilon_{\mu\nu}G^{\mu\nu}, \quad L^{-1} = I_4 - \frac{1}{2}\epsilon_{\mu\nu}G^{\mu\nu},$$

where $G^{\mu\nu}$ are the **generators** of the spinor representation of the Lorentz group:

$$G^{\mu\nu} = \frac{1}{4}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu).$$

The matrices $L$ compose the Lorentz group in spinor representation.

# Spinors

A **spinor** $\psi$ and its adjoint $\bar{\psi}$ are defined as quantities that transform according to

$$\tilde{\psi} = L\psi, \quad \tilde{\bar{\psi}} = \bar{\psi}L^{-1}.$$

Accordingly, the product $\bar{\psi}\psi$ is a scalar: $\tilde{\bar{\psi}}\tilde{\psi} = \bar{\psi}\psi$. The transformation law of the Dirac matrices shows that they can be regarded as quantities that have, in addition to the Lorentz vector index $\mu$, one spinor index and one adjoint-spinor index. The product $\psi\bar{\psi}$ transforms like the Dirac matrices: $\tilde{\psi}\tilde{\bar{\psi}} = L\psi\bar{\psi}L^{-1}$.

The spinors $\psi$ and $\bar{\psi}$ can be used to construct bilinear forms that are linear both in $\psi$ and $\bar{\psi}$ and transform like tensors. For example, $\bar{\psi}\gamma^{\mu}\psi$ transforms like a contravariant Lorentz vector: $\bar{\psi}\gamma^{\mu}\psi \to \bar{\psi}L^{-1}\Lambda^{\mu}_{\nu}L\gamma^{\nu}L^{-1}L\psi = \Lambda^{\mu}_{\nu}\bar{\psi}\gamma^{\nu}\psi$.

For an infinitesimal Lorentz transformation, the Hermitian conjugate of $L$ is $L^{\dagger} = I_4 + (1/8)\epsilon_{\mu\nu}(\gamma^{\nu\dagger}\gamma^{\mu\dagger} - \gamma^{\mu\dagger}\gamma^{\nu\dagger})$. The relations $\gamma^{0\dagger} = \gamma^0$ and $\gamma^{\alpha\dagger} = -\gamma^{\alpha}$, where $\alpha \in \{1, 2, 3\}$, together with the definition of $\gamma^{\mu}$ give $L^{\dagger}\gamma^0 = \gamma^0 L^{-1}$. Therefore, the quantity $\psi^{\dagger}\gamma^0$ transforms like an adjoint spinor: $\psi^{\dagger}\gamma^0 \to \psi^{\dagger}L^{\dagger}\gamma^0 = \psi^{\dagger}\gamma^0 L^{-1}$. Accordingly, we can associate the adjoint and the conjugate of a spinor:

$$\bar{\psi} = \psi^{\dagger}\gamma^0.$$

# Covariant derivative of spinor

The derivative of a spinor does not transform like a spinor: $\partial_i \tilde{\psi} = L\partial_i \psi + \partial_i L \psi$. Introducing the **spinor connection** $\Gamma_i$ that transforms according to

$$\tilde{\Gamma}_i = L\Gamma_i L^{-1} + \partial_i L L^{-1}$$

allows to construct the covariant derivative of a spinor:

$$\nabla_i \psi = \partial_i \psi - \Gamma_i \psi, \tag{3}$$

which transforms like a spinor: $\nabla_i \tilde{\psi} = \partial_i \tilde{\psi} - \tilde{\Gamma}_i \tilde{\psi} = L\partial_i \psi + \partial_i L \psi - (L\Gamma_i L^{-1} + \partial_i L L^{-1}) L\psi = L\nabla_i \psi$. Because $\bar{\psi}\psi$ is a scalar, $\nabla_i(\bar{\psi}\psi) = \partial_i(\bar{\psi}\psi)$, the chain rule for covariant differentiation gives the covariant derivative of an adjoint spinor:

$$\nabla_i \bar{\psi} = \partial_i \bar{\psi} + \bar{\psi}\Gamma_i. \tag{4}$$

The Dirac matrices $\gamma^\mu$ transform like $\psi\bar{\psi}$, whose covariant derivative is $\nabla_i(\psi\bar{\psi}) = \nabla_i \psi \bar{\psi} + \psi \nabla_i \bar{\psi} = \partial_i(\psi\bar{\psi}) - \Gamma_i \psi\bar{\psi} + \psi\bar{\psi}\Gamma_i = \partial_i(\psi\bar{\psi}) - [\Gamma_i, \psi\bar{\psi}]$. Therefore, the covariant derivative of a Dirac matrix is

$$\nabla_i \gamma^\mu = \omega^\mu{}_{\nu i}\gamma^\nu - [\Gamma_i, \gamma^\mu].$$

Quantities $\bar{\psi}\gamma^i \nabla_i \psi$ and $\nabla_i \bar{\psi}\gamma^i \psi$, where $\gamma^i = e^i_\mu \gamma^\mu$, transform under Lorentz rotations (and general coordinate transformations) like scalars. Their difference is imaginary.

# Fock-Ivanenko coefficients

The relation $\nabla_i \eta_{\mu\nu} = 0$ gives $\nabla_i \gamma^\mu = 0$ because the Dirac matrices $\gamma^\mu$ depend only on the tensor $\eta_{\mu\nu}$. Consequently, $\gamma_\mu \nabla_i \gamma^\mu = \omega_{\mu\nu i}\gamma^\mu\gamma^\nu - \gamma_\mu \Gamma_i \gamma^\mu + 4\Gamma_i = 0$. Its solution is $\Gamma_i = -\frac{1}{4}\omega_{\mu\nu i}\gamma^\mu\gamma^\nu - A_i I_4$, where $A_i$ is a covariant vector, which could be proportional to the electromagnetic potential. Therefore, the spinor connection $\Gamma_i$ is given, up to the addition of an arbitrary vector multiple of the unit matrix, by the Fock–Ivanenko coefficients:

$$\Gamma_i = -\frac{1}{4}\omega_{\mu\nu i}\gamma^\mu\gamma^\nu = -\frac{1}{2}\omega_{\mu\nu i}G^{\mu\nu}. \tag{5}$$

The curvature tensor with two Lorentz and two coordinate indices depends on the spin connection and its partial derivatives:

$$R^\mu{}_{\nu ik} = \omega^\mu{}_{\nu k,i} - \omega^\mu{}_{\nu i,k} + \omega^\rho{}_{\nu k}\omega^\mu{}_{\rho i} - \omega^\rho{}_{\nu i}\omega^\mu{}_{\rho k}.$$

The commutator of the covariant derivatives of a spinor is

$$\nabla_i\nabla_j\psi - \nabla_j\nabla_i\psi = K_{ij}\psi + 2S^k{}_{ij}\nabla_k\psi,$$

where

$$K_{ij} = \partial_j\Gamma_i - \partial_i\Gamma_j + [\Gamma_i, \Gamma_j] = \frac{1}{4}R_{\mu\nu ij}\gamma^\mu\gamma^\nu = \frac{1}{2}R_{\mu\nu ij}G^{\mu\nu}$$

is the curvature spinor, which transforms like a quantity with one spinor index and one adjoint-spinor index: $\tilde{K}_{ij} = LK_{ij}L^{-1}$.

# Dirac equation in curved spacetime

The Dirac Lagrangian density for a spinor field, representing a particle in a gravitational field, is given by

$$\mathcal{L}_\psi = \frac{1}{2} i\hbar e e^i_\mu (\bar{\psi}\gamma^\mu \nabla_i \psi - \nabla_i \bar{\psi}\gamma^\mu \psi) - mce\bar{\psi}\psi, \tag{6}$$

where $e$ is the determinant of the tetrad $e^i_\mu$. Varying the Lagrangian density with respect to $\bar{\psi}$ and $\psi$, respectively, gives the Dirac equation and its adjoint:

$$i\hbar\gamma^\mu e^i_\mu \nabla_i \psi = mc\psi, \quad -i\hbar\nabla_i\bar{\psi}\gamma^\mu e^i_\mu = mc\bar{\psi}. \tag{7}$$

These equations include covariant derivatives and generalize the Dirac equation to a curved spacetime.

Subtracting the first equation in multiplied by $\bar{\psi}$ from the second equation multiplied by $\psi$ gives the conservation law for the Dirac current density $j^k = e\bar{\psi}\gamma^k\psi$:

$$\bar{\psi}\gamma^k\nabla_k\psi + \nabla_k\bar{\psi}\gamma^k\psi = \nabla_k(\bar{\psi}\gamma^k\psi) = \frac{1}{e}\nabla_k(e\bar{\psi}\gamma^k\psi) = 0.$$

# Energy-momentum and spin densities

Varying the Lagrangian density with respect to the spin connection gives the spin density $\mathcal{S}_{\mu\nu}{}^{i} = -\mathcal{S}_{\nu\mu}{}^{i} = 2\delta\mathcal{L}_\psi/\delta\omega^{\mu\nu}{}_{i}$ for a spinor field:

$$\mathcal{S}_{\mu\nu}{}^{i} = \frac{1}{8}i\hbar e\bar{\psi}\{\gamma^i, [\gamma_\mu, \gamma_\nu]\}\psi = \frac{1}{2}i\hbar e\bar{\psi}\{\gamma^i, G_{\mu\nu}\}\psi. \tag{8}$$

This quantity does not depend on the spinor mass. The spin tensor is equal to the spin density divided by $e$.

Varying the Lagrangian density with respect to the tetrad gives the tetrad (canonical) energy–momentum density $\mathcal{T}_i{}^{\mu} = \delta\mathcal{L}_\psi/\delta e^i_\mu$ for a spinor field:

$$\mathcal{T}_i{}^{\mu} = \frac{1}{2}i\hbar e(\bar{\psi}\gamma^\mu\nabla_i\psi - \nabla_i\bar{\psi}\gamma^\mu\psi - e^\mu_i\bar{\psi}\gamma^j\nabla_j\psi$$
$$+e^\mu_i\nabla_j\bar{\psi}\gamma^j\psi) + mcee^\mu_i\bar{\psi}\psi. \tag{9}$$

The energy–momentum tensor is equal to the energy–momentum density divided by $e$.

# Conservation laws

The conservation law for spin density is

$$\nabla_k \mathcal{S}_{ij}{}^k - 2S_k \mathcal{S}_{ij}{}^k = \mathcal{T}_{ij} - \mathcal{T}_{ji}, \tag{10}$$

where $S_k = S^i{}_{ki}$. Without spin, the energy–momentum tensor is symmetric, as in general relativity.

The conservation law for tetrad energy–momentum density is

$$\nabla_j \mathcal{T}_i{}^j - 2S_j \mathcal{T}_i{}^j = 2S^j{}_{ki} \mathcal{T}_j{}^k + \frac{1}{2} R^{kl}{}_{ji} \mathcal{S}_{kl}{}^j. \tag{11}$$

The conservation law for the energy–momentum density gives the Dirac equation.

These conservation laws follow from the invariance of the action $(1/c)\int \mathcal{L} d\Omega$ under infinitesimal Lorentz rotations and translations, respectively.

N. Popławski, *Classical Physics: Spacetime and Fields*, arXiv:0911.0334.

# Einstein-Cartan theory

The Einstein–Cartan–Sciama–Kibble theory is the simplest theory of gravity that extends general relativity by relaxing the symmetry condition of the affine connection. In this theory, the Lagrangian density for the gravitational field is given by $\mathcal{L}_g = (-1/2\kappa)eR$, where $R = R_{ik}g^{ik}$ is the Ricci scalar, $R_{ik} = R^j{}_{ijk}$ is the Ricci tensor, and $\kappa = 8\pi G/c^4$. In general relativity, the curvature tensor $R^l{}_{ijk}$ reduces to the Riemann tensor (curvature tensor of Levi-Civita connection) $\mathring{R}^l{}_{ijk}$.

The field equations of the Einstein–Cartan theory are obtained from the principle of least action with the variations of the metric and torsion tensors. Equivalently, the field equations can be derived from the variations of the tetrad and spin connection. Equaling the variation of the total Lagrangian density for the gravitational field and matter with respect to the spin connection to zero gives the Cartan equations:

$$S^i{}_{\mu\nu} - S_\mu e^i_\nu + S_\nu e^i_\mu = -\frac{\kappa}{2e}\mathcal{S}_{\mu\nu}{}^i. \qquad (12)$$

Equaling the variation of the total Lagrangian density for the gravitational field and matter with respect to the tetrad to zero gives the Einstein equations:

$$R^\mu{}_i - \frac{1}{2}Re^\mu_i = \frac{\kappa}{e}\mathcal{T}_i{}^\mu.$$

# Einstein-Cartan theory

The Bianchi identity

$$\nabla_{[l} R^{i}{}_{|n|jk]} = 2R^{i}{}_{nm[j} S^{m}{}_{kl]}$$

and the cyclic identity

$$R^{m}{}_{[jkl]} = -2\nabla_{[l} S^{m}{}_{jk]} + 4S^{m}{}_{n[j} S^{n}{}_{kl]},$$

where the square brackets denote antisymmetrization of indices except those between the bars, together with the Einstein and Cartan field equations give the conservation laws for the spin density and for the tetrad energy–momentum density, respectively.

The field equations relate the curvature of spacetime to the energy and momentum of matter and the torsion of spacetime to the spin angular momentum of matter. According to the Cartan equations, the torsion tensor is proportional to the spin density. In the absence of spin, the torsion tensor is therefore zero and the affine connection reduces to the Levi-Civita connection, given by the Christoffel symbols. In this case, the Einstein–Cartan theory reduces general relativity. This theory is also indistinguishable in predictions from general relativity at densities of matter that are lower than the Cartan density ($\sim 10^{45}\mathrm{kg/m^3}$), so it passes all observational and experimental tests of general relativity.

# Einstein-Cartan theory

Extending general relativity to the Einstein–Cartan theory may solve several problems in quantum field theory and cosmology.

Torsion imposes a spatial extension on fermions and removes the ultraviolet divergence of radiative corrections represented by loop Feynman diagrams. The four-momentum operators in quantum mechanics in the presence of torsion do not commute. Consequently, the integration over four-momentum must be replaced by the summation over discrete four-momentum eigenvalues. Divergent integrals are replaced by convergent sums. Renormalization is finite.

N. J. Popławski, Phys. Lett. B 690, 73 (2010).

N. Popławski, Found. Phys. 50, 900 (2020).

Torsion also generates a negative correction from the spin-torsion coupling to the energy density, which acts like gravitational repulsion, which may prevent the formation of singularities in black holes and at the beginning of the Universe. Consequently, the collapsing matter in a black hole would avoid a singularity and instead reach a nonsingular bounce, after which it would expand as a new, closed universe on the other side of its event horizon. Accordingly, our Universe might have originated as a baby universe in a parent black hole existing in another universe.

N. J. Popławski, Phys. Lett. B 694, 181 (2010).

N. Popławski, Astrophys. J. 832, 96 (2016).

N. Popławski, J. Exp. Theor. Phys. 132, 374 (2021).

# Four-velocity of spinor

In this presentation, we aim to demonstrate that a relativistic spinor wave, described by the Dirac equation, can be associated with a particle of the same mass according to

$$u^i = \frac{\bar{\psi}\gamma^i\psi}{\bar{\psi}\psi}, \tag{13}$$

where $u^i = dx^i/ds$ is the four-velocity of the particle and $s$ is the affine parameter along a world line of the particle. This relation describes the **relativistic wave–particle duality**. It is equal to the ratio of the Dirac density four-current and the Dirac density scalar. In the de Broglie–Bohm pilot-wave interpretation of quantum mechanics, this relation coincides with the special-relativistic four-velocity of a particle guided by the wave function.

F. R. B. Guedes and N. J. Popławski, arXiv:2211.03234.

# Four-velocity of plane-wave spinor

It is straightforward to demonstrate that the proposed four-velocity relation is satisfied for a free particle associated with a plane spinor wave in a flat spacetime (in the absence of torsion). This four-velocity is a normalized timelike vector, which can be used to define de Broglie–Bohm trajectories for a spin-1/2 particle. This normalization also holds in curved spacetime because at the location of the particle one can construct a locally flat system of coordinates and because the normalization is a scalar relation independent of the choice of the coordinates.

The Dirac equation can be written in the Hamiltonian form:

$$i\hbar\frac{\partial\psi}{\partial t} = -i\hbar\alpha\cdot\nabla\psi + mc\beta\psi,$$

where $\alpha$ is the vector formed from the matrices $\alpha^\mu = \beta\gamma^\mu$ and $\beta = \gamma^0$. If the particle has four-momentum $p_\mu$, the corresponding spinor wave function has a form of a plane wave proportional to $\exp(-ip_\mu x^\mu/\hbar)$, where $x^\mu$ are the spacetime coordinates. Consequently, the Dirac equation becomes

$$E\psi = \mathbf{p}\cdot\alpha c\psi + mc^2\beta\psi,$$

where $E = mc^2\gamma$ is the energy, with $\gamma = (1 - v^2/c^2)^{-1/2}$, and $\mathbf{p} = m\mathbf{v}\gamma$ is the momentum of the particle, satisfying $E^2 = (\mathbf{p}c)^2 + (mc^2)^2$.

# Four-velocity of plane-wave spinor

The normalized plane-wave solution of the Dirac equation is

$$\psi(\mathbf{r},t) = \frac{1}{\sqrt{2mc^2(E+mc^2)}} \begin{pmatrix} (E+mc^2)I_2 & \sigma\cdot\mathbf{p}c \\ \sigma\cdot\mathbf{p}c & (E+mc^2)I_2 \end{pmatrix} \times \begin{pmatrix} \xi \\ \eta \end{pmatrix} \exp[i(\mathbf{p}\cdot\mathbf{r}-Et)/\hbar],$$

where $\sigma$ is the vector formed from the Pauli matrices, $I_2$ is the two-dimensional unit matrix, the square matrix is the spinor representation of the boost from rest to velocity $\mathbf{v} = \mathbf{p}c^2/E$, and $\xi$ and $\eta$ are two-dimensional (up and down) spinors describing positive (particle) and negative (antiparticle) energy states. For a spin-up, positive-energy state, $\xi = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\eta = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

Similar calculations can be carried out for a spin-down or a negative-energy state.

# Four-velocity of plane-wave spinor

The scalar bilinear composed from the plane wave is

$$\bar{\psi}\psi = \psi^\dagger\gamma^0\psi = \frac{1}{2mc^2(E+mc^2)}\Big[(E+mc^2)^2$$
$$-(1,0)(\sigma\cdot\mathbf{p})(\sigma\cdot\mathbf{p})c^2\begin{pmatrix}1\\0\end{pmatrix}\Big] = \frac{(E+mc^2)^2-\mathbf{p}^2c^2}{2mc^2(E+mc^2)} = 1.$$

The vector bilinear components composed from the plane wave, using the relation $p^\mu = mcu^\mu$ for a free particle and an identity $\sigma^\mu\sigma^\nu+\sigma^\nu\sigma^\mu = -2\eta^{\mu\nu}I_2$, are

$$\bar{\psi}\gamma^0\psi = \psi^\dagger\psi = \frac{1}{2mc^2(E+mc^2)}\Big[(E+mc^2)^2$$
$$+(1,0)(\sigma\cdot\mathbf{p})(\sigma\cdot\mathbf{p})c^2\begin{pmatrix}1\\0\end{pmatrix}\Big] = \frac{(E+mc^2)^2+\mathbf{p}^2c^2}{2mc^2(E+mc^2)} = \frac{E}{mc^2} = u^0,$$

and

$$\bar{\psi}\gamma^\mu\psi = \psi^\dagger\alpha^\mu\psi = \frac{E+mc^2}{2mc^2(E+mc^2)}\Big[(1,0)\sigma^\mu(\sigma\cdot\mathbf{p})c$$
$$\times\begin{pmatrix}1\\0\end{pmatrix} + (1,0)(\sigma\cdot\mathbf{p})c\sigma^\mu\begin{pmatrix}1\\0\end{pmatrix}\Big] = \frac{p^\mu}{mc} = u^\mu.$$

# Momentum and spin of spinor

The conservation of the momentum four-vector $P_i$ follows from the symmetry of a system under spacetime translations. The four-momentum operator $\hat{P}_i$ of a spinor is therefore associated with a generator of translation, which in curved spacetime is given by a covariant derivative:

$$\hat{P}_i\psi = i\hbar\nabla_i\psi, \quad \hat{P}_i\bar{\psi} = -i\hbar\nabla_i\bar{\psi}. \tag{14}$$

The conservation of the intrinsic angular momentum (spin) four-tensor $S_{ik}$ follows from the symmetry of a system under spacetime rotations. The spin four-tensor operator $\hat{S}_{ik}$ of a spinor is therefore associated with a generator of rotation $G_{ik}$:

$$\hat{S}_{ik}\psi = i\hbar G_{ik}\psi, \quad \hat{S}_{ik}\bar{\psi} = i\hbar\bar{\psi}G_{ik}, \tag{15}$$

following $\hat{S}_{ik}\bar{\psi} = \hat{S}_{ik}\psi^\dagger\gamma^0 = (iG_{ik}\psi)^\dagger\gamma^0 = -i\psi^\dagger G^\dagger_{ik}\gamma^0 = i\psi^\dagger\gamma^0 G_{ik} = i\bar{\psi}G_{ik}$.

For free (without fields other than gravity) particle with definite values of the four-momentum and spin, which are respectively eigenvalues of their operators, the operators can be replaced by the eigenvalues. The mass $m$ of the particle is given by

$$mc = u^i P_i.$$

The Dirac equations for a spinor and its adjoint can be written as $P_i\gamma^i\psi = mc\psi$ and $\bar{\psi}\gamma^i P_i = mc\bar{\psi}$, giving $P_i\frac{\bar{\psi}\gamma^i\psi}{\bar{\psi}\psi} = mc$, in agreement with the proposed spinor four-velocity.

# Momentum and spin of spinor

Substituting the four-momentum of a spinor into the tetrad energy–momentum density gives

$$\mathcal{T}_i{}^\mu = e(P_i\bar{\psi}\gamma^\mu\psi - e_i^\mu P_j\bar{\psi}\gamma^j\psi) + mcee_i^\mu\bar{\psi}\psi.$$

Using the spinor four-velocity, this density becomes

$$\mathcal{T}_i{}^\mu = e\bar{\psi}\psi(P_i u^\mu - e_i^\mu P_j u^j) + mcee_i^\mu\bar{\psi}\psi = e\bar{\psi}\psi P_i u^\mu,$$
$$\mathcal{T}_i{}^k = e\bar{\psi}\psi P_i u^k. \quad (16)$$

Substituting the four-spin of a spinor into the spin density gives

$$\mathcal{S}_{\mu\nu}{}^i = \frac{1}{2}i\hbar e\bar{\psi}(\gamma^i G_{\mu\nu} + G_{\mu\nu}\gamma^i)\psi = e\bar{\psi}\gamma^i\psi S_{\mu\nu}.$$

Using the spinor four-velocity, this density becomes

$$\mathcal{S}_{jk}{}^i = e\bar{\psi}\psi S_{jk}u^i. \quad (17)$$

The generators satisfy $\gamma^\nu G_{\mu\nu} = 0$. Contracting the four-spin with the four-velocity gives $S_{ik}u^k = 0$. The spin four-tensor is orthogonal to the four-velocity.

# Equations of motion for spinor

The orthogonality of the four-spin and four-velocity gives ${\mathcal{S}_{jk}}^k = 0$. Consequently, the Cartan equations give

$$S_i = 0.$$

The conservation law for spin density with gives

$$\nabla_k(e\bar{\psi}\psi S_{ij}u^k) = e\bar{\psi}\psi(P_i u_j - P_j u_i).$$

Using $u^k\nabla_k = D/ds$, this relation can be written as

$$e\bar{\psi}\psi(P_i u_j - P_j u_i) = e\bar{\psi}\psi u^k\nabla_k S_{ij} + S_{ij}\nabla_k(e\bar{\psi}\psi u^k)$$
$$= e\bar{\psi}\psi\frac{DS_{ij}}{ds} + S_{ij}\nabla_k(e\bar{\psi}\gamma^k\psi) = e\bar{\psi}\psi\frac{DS_{ij}}{ds},$$

which is equivalent to

$$\frac{DS^{ij}}{ds} = P^i u^j - P^j u^i. \qquad (18)$$

This equation is the classical Mathisson–Papapetrou equation of motion for the four-spin of a particle.

# Equations of motion for spinor

Using the conservation law for tetrad energy–momentum density gives

$$\nabla_j(e\bar{\psi}\psi P_i u^j) = 2S^j{}_{ki}(e\bar{\psi}\psi P_j u^k) + \frac{1}{2}R^{kl}{}_{ji}(e\bar{\psi}\psi S_{kl}u^j).$$

This relation can be written as

$$2S^j{}_{ki}(e\bar{\psi}\psi P_j u^k) + \frac{1}{2}R^{kl}{}_{ji}(e\bar{\psi}\psi S_{kl}u^j) = e\bar{\psi}\psi u^j\nabla_j P_i$$
$$+P_i\nabla_j(e\bar{\psi}\psi u^j) = e\bar{\psi}\psi\frac{DP_i}{ds} + P_i\nabla_j(e\bar{\psi}\gamma^j\psi) = e\bar{\psi}\psi\frac{DP_i}{ds},$$

which is equivalent to

$$\frac{DP^i}{ds} = 2S_{jk}{}^i P^j u^k + \frac{1}{2}R_{jkl}{}^i S^{jk}u^l. \qquad (19)$$

This equation is the classical Mathisson–Papapetrou equation of motion for the four-momentum of a particle.

Consequently, if a spinor satisfies the Dirac equation, then the corresponding particle with four-momentum eigenvalue $P_i$, spin four-tensor eigenvalue $S_{ik}$, and four-velocity $u^i = \frac{\bar{\psi}\gamma^u\psi}{\bar{\psi}\psi}$ satisfies the Mathisson–Papapetrou equations.

They can also be derived from the covariant Heisenberg equation for operators, extending to torsion: S. K. Wong, Int. J. Theor. Phys. 5, 221 (1972).

# Antisymmetry of spin tensor

The spin density for a spinor field is equivalent to

$$\mathcal{S}^{ijk} = \frac{1}{2}i\hbar\, e\bar{\psi}\gamma^{[i}\gamma^{j}\gamma^{k]}\psi = \mathcal{S}^{[ijk]}, \tag{20}$$

which is completely antisymmetric, leading to

$$S_{jk}u_i = -S_{ji}u_k.$$

Contracting this relation with $u^i$ and using the orthogonality of the four-spin and fur-velocity yields

$$S_{jk} = -S_{ji}u^i u_k = 0, \quad S_{ij} = 0.$$

This result mean that the pole approximation of a particle must be extended to the next-order pole-dipole approximation in order to account for the spin angular momentum in the equations of motion.

In the pole approximation, the Mathisson–Papapetrou equations reduce to the geodesic equation of motion. Consequently, if a spinor satisfies the Dirac equation, then the corresponding particle moves along the geodesic. This relation constitutes the relativistic wave–particle duality: a wave propagating in a curved spacetime guides the corresponding particle in a way that is equivalent to the motion of the particle described by general relativity.

Acknowledgments:

University Research Scholar program at University of New Haven

Gabriel Unger, Jordan Cubero, Michael Del Grosso, Francisco Guedes

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